Nonlinear Analysis 224 (2022) 113056

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Variational and stability properties of coupled NLS equations on the star graph

Liliana Cely, Nataliia Goloshchapova *

Institute of Mathematics and Statistics, University of São Paulo, Rua do Matão, 1010, São Paulo-SP, 05508-090, Brazil

ARTICLE INFO

Article history:
Received 8 January 2022
Accepted 22 June 2022
Communicated by Gustavo Ponce

MSC:
primary 35Q55
secondary 35Q40

Keywords:
δ coupling
Ground state
Nonlinear Schrödinger equation
Orbital stability

ABSTRACT

We consider variational and stability properties of a system of two coupled nonlinear Schrödinger equations on the star graph Γ with the δ coupling at the vertex of Γ. The first part is devoted to the proof of an existence of the ground state as the minimizer of the constrained energy in the cubic case. This result extends the one obtained recently for the coupled NLS equations on the line.

In the second part, we study stability properties of several families of standing waves in the case of a general power nonlinearity. In particular, we consider one-component standing waves $e^{i\omega t}(\Phi_1(x), 0)$ and $e^{i\omega t}(0, \Phi_2(x))$. Moreover, we study two-component standing waves $e^{i\omega t}(\Phi(x), \Phi(x))$ for the case of power nonlinearity depending on a unique power parameter p.

To our knowledge, these are the first results on variational and stability properties of coupled NLS equations on graphs.

1. Introduction

The nonlinear Schrödinger equation with the focusing power nonlinearity and the δ coupling on the star graph Γ

$$i\partial_t u(t,x) + \Delta_\gamma u(t,x) + |u(t,x)|^{q-2} u(t,x) = 0 \tag{1.1}$$

has been extensively studied during the last decade (see [1,4,6,20,26,42]). Here $\gamma \in \mathbb{R}\setminus\{0\}$, $q > 2$, $u : \mathbb{R}\times\Gamma \to \mathbb{C}^N$, and Δ_γ is the Laplace operator on $L^2(\Gamma)$ with the δ coupling: for $v = (v_e)_{e=1}^N$

$$(-\Delta_\gamma v)(x) = (-v_e''(x))_{e=1}^N, \qquad \mathrm{dom}(-\Delta_\gamma) = \left\{ v \in H^2(\Gamma) : \sum_{e=1}^N v_e'(0) = -\gamma v_1(0) \right\}.$$

The well-posedness of (1.1) was established in [3,8,20], whereas the existence and the stability of standing waves were studied in [2,3,6,7,26].

* Corresponding author.
E-mail addresses: mlcelyp@ufmg.br (L. Cely), nataliia@ime.usp.br (N. Goloshchapova).

https://doi.org/10.1016/j.na.2022.113056

Nonlinear PDEs on graphs appear as mathematical models which describe various physical phenomena. In particular, (1.1) appears as a preferred model in optics of nonlinear Kerr media and dynamics of Bose–Einstein condensates (see [10,11,25,42]). Graph models arise as an approximation of multidimensional narrow waveguides when their thickness parameters converge to zero (see [16,18,29,39,46]).

In the first part of this paper we consider two coupled nonlinear Schrödinger equations

$$\begin{cases} i\partial_t u(t,x) + \Delta_\gamma u(t,x) + \left(a\,|u(t,x)|^2 + b\,|v(t,x)|^2\right) u(t,x) = 0 \\ i\partial_t v(t,x) + \Delta_\gamma v(t,x) + \left(b\,|u(t,x)|^2 + c\,|v(t,x)|^2\right) v(t,x) = 0 \\ (u(0,x), v(0,x)) = (u_0(x), v_0(x)), \end{cases} \tag{1.2}$$

where $\gamma > 0$, $a, b, c \in \mathbb{R}$, $(t,x) \in \mathbb{R}\times\Gamma$, $u, v : \mathbb{R}\times\Gamma \to \mathbb{C}^N$. Coupled NLS equations appear in many physical applications: interaction of waves with different polarizations, description of nonlinear modulations of two monochromatic waves, interaction of Bloch-wave packets in a periodic system, evolution of two orthogonal pulse envelopes in birefringent optical fiber, evolution of two surface wave packets in deep water, the Hartree–Fock theory for a double condensate (for the references see [5,12,28,37,38,47]).

From the physical and the mathematical point of view, an interesting issue is to study the existence and the stability of standing waves of system (1.2). For the principal results on the existence of solitary waves for the coupled NLS equations on $\mathbb{R}$ and $\mathbb{R}^d$ and their stability/instability properties, the reader is addressed to [9,15,33–36,41,43,44,48]. *A standing wave solution* of (1.2) is a solution of the form $\left(e^{i\omega_1 t}\Phi_1(x), e^{i\omega_2 t}\Phi_2(x)\right)$, where $\omega_1, \omega_2 \in \mathbb{R}$ and (Φ_1, Φ_2) solves the following stationary problem

$$\begin{cases} -\Delta_\gamma \Phi_1 + \omega_1 \Phi_1 - \left(a\,|\Phi_1|^2 + b\,|\Phi_2|^2\right)\Phi_1 = 0 \\ -\Delta_\gamma \Phi_2 + \omega_2 \Phi_2 - \left(c\,|\Phi_2|^2 + b\,|\Phi_1|^2\right)\Phi_2 = 0. \end{cases} \tag{1.3}$$

It is classical idea to look for the profile of the solitary wave of a Hamiltonian system as a solution of a certain minimization problem. In Section 2 we study the existence of the profiles $(\Phi_1(x), \Phi_2(x))$ being minimizers of the energy under the fixed mass constraint (depending on the constants a, b, c, γ). In particular, we find explicitly the minimizer $(e^{i\theta_1}\Phi_1(x), e^{i\theta_2}\Phi_2(x))$, $\theta_1, \theta_2 \in \mathbb{R}$, where (Φ_1, Φ_2) is the solution to the stationary problem

$$\begin{cases} -\Delta_\gamma \Phi_1 + \omega \Phi_1 - \frac{b^2-ac}{b-c}|\Phi_1|^2 \Phi_1 = 0 \\ -\Delta_\gamma \Phi_2 + \omega \Phi_2 - \frac{b^2-ac}{b-a}|\Phi_2|^2 \Phi_2 = 0. \end{cases}$$

We managed to adapt the method of [41] (elaborated for the coupled NLS equations on the line) for the case of the star graph. This method requires that the constants a, b, c satisfy one of the following two assumptions:

$(A1)$ $0 < b < \min\{a, c\}$ or

$(A2)$ $a, c > 0$, $b > \max\{a, c\}$.

The main idea is to use the concentration-compactness principle [31,32] and the technique of symmetric rearrangements (which is used to prove the absence of runaway case). The concentration-compactness principle for the star graph was elaborated in [2] and extended in [13] to the case of a general starlike graph. While the technique of symmetric rearrangements for the star graph has been introduced in [3]. Moreover, we exploit the existence end the explicit form of the minimizer of the constrained energy for the unique NLS equation with the δ coupling on Γ obtained in [2]. The orbital stability of the standing wave associated with the minimizer follows standardly (see Section 2.3).

Notice that the results by [41] were recently extended in [12] for the generalized power nonlinearity

$$F_{p,q,r}(u,v) = (a|u|^{q-2}u + b|v|^p|u|^{p-2}u, c|v|^{r-2}v + b|u|^p|v|^{p-2}v).$$

It seems much more difficult to extend the technique from [12] for the case of the star graph. In particular, it is not clear how to prove that the corresponding variational problem is subadditive. The main difficulty is the presence of the term $\frac{N}{2}$ in Pólya–Szegő inequality (A.29).

In the second part of the paper (see Section 3) we deal with the stability properties of the standing waves for the system of coupled NLS equations on Γ. In particular, in Section 3.1 we study one-component (one-hump) standing waves $(e^{i\omega t}\Phi_1(x), 0)$ and $(0, e^{i\omega t}\Phi_2(x))$ and the nonlinearity $F_{p,q,r}(u,v)$. The profiles Φ_1 and Φ_2 satisfy the following stationary equations

$$-\Delta_\gamma \Phi_1 + \omega\Phi_1 - a|\Phi_1|^{q-2}\Phi_1 = 0, \quad -\Delta_\gamma \Phi_2 + \omega\Phi_2 - c|\Phi_2|^{r-2}\Phi_2 = 0. \tag{1.4}$$

The description of solutions to each equation in (1.4) was obtained in [3, Theorem 4]. Namely, solutions constitute a family of $\left[\frac{N-1}{2}\right]+1$ vector-functions (one symmetric and the rest asymmetric). To prove spectral instability results for the family we use an abstract result from [21] (which permits to estimate the number of unstable eigenvalues $\lambda > 0$). For $p, q > 3$, using C^2 regularity of the mapping data-solution (associated with the corresponding Cauchy problem) and applying the abstract result from [24], we have shown orbital instability for the profiles Φ_1, Φ_2. This abstract result states the nonlinear instability of a fixed point of a nonlinear mapping having the linearization L of spectral radius $r(L) > 1$. To apply the approach by [21] we need to estimate the Morse index of two self-adjoint in $L^2(\Gamma)$ operators associated with the second derivative of the action functional. These estimates were obtained in [26, Theorem 3.1].

In Section 3.2 we study two-component (multi-hump) standing waves $(e^{i\omega t}\Phi(x), e^{i\omega t}\Phi(x))$ in the case of the one-parametric power nonlinearity

$$F_p(u,v) = (|u|^{2p-2}u + b|v|^p|u|^{p-2}u, |v|^{2p-2}v + b|u|^p|v|^{p-2}v).$$

It is easily seen that $\Phi(x)$ solves

$$-\Delta_\gamma \Phi + \omega\Phi - (b+1)\Phi^{2p-1} = 0. \tag{1.5}$$

As in the previous case, the solutions form a similar family of $\left[\frac{N-1}{2}\right]+1$ vector-functions $(\Phi_k^\gamma, \Phi_k^\gamma), k = 0, \ldots, \left[\frac{N-1}{2}\right]$. Again we use the results by [21,24] mentioned above to prove spectral and orbital instability results for $\gamma > 0, k \geq 1$ and $\gamma < 0, k \geq 0$. It is worth noting that in the multi-hump case one of the main technical difficulties is that the "real" part $\tilde{L}^R$ of a self-adjoint operator associated with the second derivative of the action functional is not diagonal (as it was in the previous case). To overcome this difficulty we diagonalize the system $\tilde{L}^R\vec{h} = \lambda\vec{h},\ \vec{h} = (h_1, h_2)$ making linear transformation $h_+ = h_1 + h_2,\ h_- = h_1 - h_2$.

Separately, we prove the orbital stability result for $\gamma > 0, k = 0$ (that is, we consider the symmetric profile Φ_0^γ with decaying on $\mathbb{R}^+$ components). We generalize the approach by [34]. The key argument is to use the analytic perturbation theory to count the number of positive eigenvalues of the Hessian associated with the action functional at (ω, ω).

In Section 3.3 we consider $F_p(u,v)$ for $p = 2, b = 1$. In this situation, the system obeys 2D rotation invariance. Using [23, Stability and Instability Theorem] we prove spectral instability results and orbital stability result for a standing wave generated by $2D$ rotation (stability is due to the centralizer subgroup).

1.1. Notation and some useful facts

A star graph Γ is constructed by the union of $N \geq 2$ infinite half-lines connected at a single vertex ν. Each edge I_e, $e = 1, \ldots, N$, can be regarded as $\mathbb{R}^+$, and the vertex ν is placed at the origin. Given a function $u : \Gamma \to \mathbb{C}^N$, $u = (u_e)_{e=1}^N$, where $u_e : \mathbb{R}^+ \to \mathbb{C}$ denotes the restriction of u to I_e. In particular, the nonlinear term in (1.2) is defined componentwise:

$$\begin{cases} \left(a\left|u\right|^2+b\left|v\right|^2\right)u=\left(\left(a\left|u_e\right|^2+b\left|v_e\right|^2\right)u_e\right)_{e=1}^N \\ \left(b\left|u\right|^2+c\left|v\right|^2\right)v=\left(\left(b\left|u_e\right|^2+c\left|v_e\right|^2\right)v_e\right)_{e=1}^N. \end{cases}$$

We denote by $u_e(0)$ and $u_e'(0)$ the limits of $u_e(x)$ and $u_e'(x)$ as $x\to 0^+$. We say that a function u is *continuous* on Γ if every restriction u_e is continuous on I_e and $u_1(0)=\cdots=u_N(0)$. The space of continuous functions is denoted by $C(\Gamma)$.

The natural Hilbert space associated with the Laplace operator Δ_γ is $L^2(\Gamma)$, which is defined as $L^2(\Gamma)=\bigoplus_{e=1}^N L^2(\mathbb{R}^+)$. The inner product in $L^2(\Gamma)$ is given by

$$(u,v)_2=\operatorname{Re}\sum_{e=1}^N (u_e,v_e)_{L^2(\mathbb{R}^+)},\quad u=(u_e)_{e=1}^N, v=(v_e)_{e=1}^N.$$

Analogously, for $1\le p\le\infty$, we define the space $L^p(\Gamma)$ as the set of functions on Γ whose components belong to $L^p(\mathbb{R}^+)$, and the norm is defined by

$$\|u\|_p^p=\sum_{e=1}^N\|u_e\|^p_{L^p(\mathbb{R}^+)},\ \ p\neq\infty,\qquad \|u\|_\infty=\max_{1\le e\le N}\|u\|_\infty.$$

Depending on the context, $\|\cdot\|_p$ will denote the norm in L^p either on the graph Γ or on the line. The Sobolev spaces $H^1(\Gamma)$ and $H^2(\Gamma)$ are defined as

$$H^k(\Gamma)=\left\{u\in C(\Gamma):\ u_e\in H^1(\mathbb{R}^+),\ \ e=1,\ldots,N\right\},\ k=1,2.$$

The proof of the following proposition is a direct consequence of the Gagliardo–Nirenberg inequality on $\mathbb{R}^+$ (see [40, I.31]).

Proposition 1.1. *Let $q\in[2,\infty]$, $1\le p\le q$, and $\mu=\frac{\frac1p-\frac1q}{\frac12+\frac1p}$, then there exists $C>0$ such that*

$$\|u\|_q\le C\,\|u'\|_2^\mu\,\|u\|_p^{1-\mu} \tag{1.6}$$

for any $u\in H^1(\Gamma)$.

Remark 1.2. Observe that for a compact graph (that is, an abstract graph without infinite edges), a weaker version of the Gagliardo–Nirenberg inequality holds:

$$\|u\|_q\le C\,\|u\|_{H^1}^\mu\,\|u\|_p^{1-\mu}. \tag{1.7}$$

We define the space X to be the Cartesian product $H^1(\Gamma)\times H^1(\Gamma)$ equipped with the norm $\|(u,v)\|_X^2=\|u\|_{H^1}^2+\|v\|_{H^1}^2$, and X^* stands for its dual. The corresponding duality product is denoted by $\langle\cdot,\cdot\rangle_{X^*\times X}$.

Throughout this paper we use $C, C_\varepsilon, C_{\varepsilon,\alpha,\beta}$ to denote various positive constants whose actual value is not important and which may vary from line to line. By $n(L)$ we denote the number of negative eigenvalues of a linear operator L (counting multiplicities).

2. Variational analysis: The case of the cubic nonlinearity

2.1. Statement of the main result

The energy and the mass functionals associated to (1.2) are defined by

$$E(u,v) = \|u'\|_2^2 + \|v'\|_2^2 - \gamma\left(|u_1(0)|^2 + |v_1(0)|^2\right) - \frac{1}{2}\left(a\|u\|_4^4 + c\|u\|_4^4\right) - b\|uv\|_2^2$$

and

$$Q(u) = \|u\|_2^2, \qquad Q(v) = \|v\|_2^2. \tag{2.1}$$

Proposition 2.1. *For any $(u_0, v_0) \in X$ there exists a unique maximal solution $(u,v) \in C(\mathbb{R}, X) \cap C^1(\mathbb{R}, X^*)$ of (1.2) satisfying $(u(0,x), v(0,x)) = (u_0, v_0)$. Moreover,*

$$E(u(t), v(t)) = E(u_0, v_0), \quad Q(u(t)) = Q(u_0), \quad Q(v(t)) = Q(v_0) \quad \textit{for all } t \in \mathbb{R}.$$

The proof of the local well-posedness follows analogously to [14, Theorem 4.10.1] (see also Section 2 in [3]). For the proof of the global well-posedness see Remark 3.1.

For given real constants a, b, c satisfying either $(A1)$ or $(A2)$ (see Introduction) and fixed frequency $\omega > \frac{\gamma^2}{N^2}$ we set

$$\alpha(\omega) = \frac{2(b-c)}{b^2 - ac}(N\sqrt{\omega} - \gamma), \qquad \beta(\omega) = \frac{2(b-a)}{b^2 - ac}(N\sqrt{\omega} - \gamma).$$

It is obvious that $\alpha(\omega)$ and $\beta(\omega)$ run the interval $(0, \infty)$ when ω runs the interval $(\frac{\gamma^2}{N^2}, \infty)$.

We are interested in ground states of (1.3). By *a ground state* we mean a minimizer $(\Phi_1, \Phi_2) \in X$ of the energy E in $H^1(\Gamma)$ constrained to the manifold of the states with fixed masses $Q(\Phi_1) = \alpha(\omega)$ and $Q(\Phi_2) = \beta(\omega)$. That is, we study the constrained variational problem

$$J(\alpha, \beta) = \inf\left\{E(u,v) \,:\, u, v \in H^1(\Gamma),\ \|u\|_2^2 = \alpha(\omega),\ \|v\|_2^2 = \beta(\omega)\right\}. \tag{2.2}$$

In what follows we will use notation α, β instead of $\alpha(\omega), \beta(\omega)$. A *minimizing sequence* for (2.2) is a sequence $\{(u_n, v_n)\}$ in X such that $Q(u_n) = \alpha$, $Q(v_n) = \beta$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} E(u_n, v_n) = J(\alpha, \beta)$. We denote the set of nontrivial minimizers by

$$\mathcal{G}(\alpha, \beta) = \left\{(u,v) \in X \,:\, J(\alpha,\beta) = E(u,v),\ \|u\|_2^2 = \alpha,\ \|v\|_2^2 = \beta\right\}. \tag{2.3}$$

Below we state the main result of the first part.

Theorem 2.2. *Suppose that the numbers a, b, c satisfy either (A1) or (A2), $\omega > \frac{\gamma^2}{N^2}$, $\gamma > 0$, and $N\sqrt{\omega} - \gamma \le \frac{2\gamma}{N}$. Then the following statements hold.*
(i) Any minimizing sequence $\{(u_n, v_n)\}$ for $J(\alpha, \beta)$ is relatively compact in X. That is, there exist a subsequence $\{(u_{n_k}, v_{n_k})\}$ and $(\Phi_1, \Phi_2) \in X$ such that $\{(u_{n_k}, v_{n_k})\}$ converges to (Φ_1, Φ_2) in X. Therefore, there exists a minimizer for problem (2.2), and hence $\mathcal{G}(\alpha, \beta)$ is non-empty.
(ii) For each minimizing sequence $\{(u_n, v_n)\}$ we have:

$$\lim_{n\to\infty} \inf_{(\Phi_1, \Phi_2) \in \mathcal{G}(\alpha,\beta)} \|(u_n, v_n) - (\Phi_1, \Phi_2)\|_X = 0. \tag{2.4}$$

Remark 2.3. The restriction $N\sqrt{\omega}-\gamma\leq\frac{2\gamma}{N}$ has to be commented.

(i) The method that we use requires an existence of a minimizer for problems (2.15). Due to [2, Theorem 1] the problem

$$\inf\left\{\frac{1}{2}\left(\|u'\|_2^2-\gamma|u_1(0)|^2-\frac{1}{2}\|u\|_4^4\right): u\in H^1(\Gamma),\ \|u\|_2^2=m\right\}\tag{2.5}$$

has a solution for a small mass, i.e. $m\leq m^*=\frac{4\gamma}{N}$ (see formula (1.4) in [2]). In problems (2.15) the term $\frac{1}{2}\|u\|_4^4$ from (2.5) has to be substituted by $\frac{b^2-ac}{2(b-c)}\|u\|_4^4$ and $\frac{b^2-ac}{2(b-a)}\|u\|_4^4$. Then in this case we will have that m^* coincides with $\frac{4\gamma}{N}\frac{(b-c)}{b^2-ac}$ and $\frac{4\gamma}{N}\frac{(b-a)}{b^2-ac}$ respectively. Finally, recalling equalities $\alpha=\frac{2(b-c)}{b^2-ac}(N\sqrt{\omega}-\gamma)$, $\beta=\frac{2(b-a)}{b^2-ac}(N\sqrt{\omega}-\gamma)$, and assuming that m coincides either with α or with β, we obtain that the restriction $m\leq m^*$ is equivalent to $N\sqrt{\omega}-\gamma\leq\frac{2\gamma}{N}$.

(ii) It is interesting to compare the interval $(\frac{\gamma^2}{N^2},\frac{1}{N^2}\left(\frac{2\gamma}{N}+\gamma\right)^2]$ of existence of solution to (2.2) with the one given by [13, Theorem 3]. In particular, the theorem states that for $\omega\in(E_0,E_0+\delta)$ (with sufficiently small δ) the stationary equation

$$H\Phi+\omega\Phi-|\Phi|^{q-2}\Phi=0$$

has a unique solution. Here H is the Laplace operator with the δ coupling and a linear potential on a general starlike graph, and $E_0=-\inf\sigma(H)$.

In [3, Remark 4.1] the authors mention that the restriction $m\leq m^*$ is not optimal, and the interval of existence of solution to minimization problem (2.5) is bigger. This fact suggests that the solution to (2.2) exists in some interval $(\frac{\gamma^2}{N^2},\frac{\gamma^2}{N^2}+\delta)\supset(\frac{\gamma^2}{N^2},\frac{1}{N^2}\left(\frac{2\gamma}{N}+\gamma\right)^2]$.

We have managed to obtain the explicit characterization of the set of minimizers $\mathcal{G}(\alpha,\beta)$.

Theorem 2.4. *Suppose that assumptions of Theorem 2.2 hold. Then the set of ground states is given by*

$$\mathcal{G}(\alpha,\beta)=\left\{\left(e^{i\theta_1}\sqrt{\frac{b-c}{b^2-ac}}\Phi_{\omega,\gamma}(\cdot),e^{i\theta_2}\sqrt{\frac{b-a}{b^2-ac}}\Phi_{\omega,\gamma}(\cdot)\right)\ :\theta_1,\theta_2\in\mathbb{R}\right\},$$

where $(\Phi_{\omega,\gamma})_e=\phi_{\omega,\gamma}$, $e=1,\dots,N$, *with*

$$\phi_{\omega,\gamma}(x)=\sqrt{2\omega}\ \mathrm{sech}\left(\sqrt{\omega}x+\tanh^{-1}\left(\frac{\gamma}{N\sqrt{\omega}}\right)\right).\tag{2.6}$$

Next we give the definition of an orbital stability.

Definition 2.5. The standing wave $(u(t,x),v(t,x))=\left(e^{i\omega t}\Phi_1(x),e^{i\omega t}\Phi_2(x)\right)$ is said to be orbitally stable in X if for any $\varepsilon>0$ there exists $\eta>0$ with the following property: if $(u_0,v_0)\in X$ satisfies

$$\|(u_0,v_0)-(\Phi_1,\Phi_2)\|_X<\eta,$$

then the solution $(u(t,x),v(t,x))$ of (1.2) with $(u(0,\cdot),v(0,\cdot))=(u_0,v_0)$ satisfies

$$\sup_{t\geq 0}\inf_{\theta_1,\theta_2\in\mathbb{R}}\left\|(u(t,\cdot),v(t,\cdot))-\left(e^{i\theta_1}\Phi_1,e^{i\theta_2}\Phi_2\right)\right\|_X<\varepsilon.$$

Otherwise, the standing wave $\left(e^{i\omega t}\Phi_1(x),e^{i\omega t}\Phi_2(x)\right)$ is said to be orbitally unstable in X.

Using the arguments from [2,3,41], the compactness of the minimizing sequences (see Lemma 2.16 below), and the uniqueness of the ground state (up to phase) proved in Theorem 2.4, one arrives at the following result.

Corollary 2.6. *Suppose that assumptions of Theorem 2.2 hold. Then the standing wave solution*

$$\left(e^{i\omega t}\sqrt{\frac{b-c}{b^2-ac}}\,\Phi_{\omega,\gamma}, e^{i\omega t}\sqrt{\frac{b-a}{b^2-ac}}\,\Phi_{\omega,\gamma}\right),$$

where $\Phi_{\omega,\gamma}$ is defined by (2.6), is orbitally stable in X.

2.2. Existence of ground states

In this subsection we prove Theorem 2.2. We have divided the proof into several lemmas and propositions. Throughout this subsection, we assume that the hypotheses of Theorem 2.2 hold.

Proposition 2.7. *For any $\alpha, \beta > 0$ we have $-\infty < J(\alpha,\beta) < 0$.*

Proof. Firstly, we show $J(\alpha,\beta) < 0$. Let $u \in H^1(\Gamma)$ be such that $\|u\|_2^2 = \alpha$ and we take $v = \sqrt{\frac{\beta}{\alpha}}u$, then $\|v\|_2^2 = \beta$. Define $u_r(x) = \sqrt{r}u(rx)$ and $v_r(x) = \sqrt{r}v(rx)$ for $r > 0$. Since $\|u_r\|_2^2 = \alpha$, $\|v_r\|_2^2 = \beta$, and $\|u_r'\|_2^2 = r^2\,\|u'\|_2^2$, then we obtain

$$E(u_r, v_r) = r^2\left(\|u'\|_2^2 + \|v'\|_2^2\right) - r\gamma\left(|u_1(0)|^2 + |v_1(0)|^2\right) - \frac{r}{2}\left(a\|u\|_4^4 + c\|v\|_4^4\right) - rb\|uv\|_2^2.$$

Since $v = \sqrt{\frac{\beta}{\alpha}}u$, then we have

$$\begin{aligned} E(u_r, v_r) &= \left(1+\frac{\beta}{\alpha}\right)\left(r^2\,\|u'\|_2^2 - r\gamma\,|u_1(0)|^2\right) - r\frac{(b^2-ac)(2b-a-c)}{2(b-c)^2}\,\|u\|_4^4 \\ &\le \left(1+\frac{\beta}{\alpha}\right) r^2\,\|u'\|_2^2 - r\frac{(b^2-ac)(2b-a-c)}{2(b-c)^2}\,\|u\|_4^4 . \end{aligned}$$

By $(b^2-ac)(2b-a-c) > 0$, we can choose r small enough to ensure that $E(u_r, v_r) < 0$, hence $J(\alpha,\beta) < 0$. Secondly, we prove that $J(\alpha,\beta) > -\infty$. Using Gagliardo–Nirenberg inequality (1.6), the Cauchy–Schwarz inequality, and the Young inequality, we have

$$|u_1(0)|^2 \le \|u\|_\infty^2 \le C\,\|u'\|_2\,\|u\|_2 \le \varepsilon\|u'\|_2^2 + C_\varepsilon\|u\|_2^2, \tag{2.7}$$

$$\int_\Gamma |u|^2\,|v|^2\,dx \le \|u\|_4^2\,\|v\|_4^2 \le \frac{1}{2}\left(\|u\|_4^4 + \|v\|_4^4\right), \tag{2.8}$$

and

$$\|u\|_4^4 = C\,\|u'\|_2\,\|u\|_2^3 \le \varepsilon\,\|u'\|_2^2 + C_\varepsilon\,\|u\|_2^6 \tag{2.9}$$

for all $u \in H^1(\Gamma)$ and any $\varepsilon > 0$. Therefore, by (2.7), (2.8), and (2.9), we have

$$\begin{aligned} E(u,v) &= \|u'\|_2^2 + \|v'\|_2^2 - \gamma\left(|u_1(0)|^2 + |v_1(0)|^2\right) - \frac{1}{2}\left(a\,\|u\|_4^4 + c\,\|v\|_4^4\right) - b\|uv\|_2^2 \\ &\ge \|u'\|_2^2 + \|v'\|_2^2 - \gamma\left(|u_1(0)|^2 + |v_1(0)|^2\right) - C\left(\|u\|_4^4 + \|v\|_4^4\right) \\ &\ge (1-\varepsilon)\left(\|u'\|_2^2 + \|v'\|_2^2\right) - C_\varepsilon\left(\|u\|_2^2 + \|v\|_2^2 + \|u\|_2^6 + \|v\|_2^6\right) \\ &= (1-\varepsilon)\left(\|u'\|_{H^1}^2 + \|v'\|_{H^1}^2\right) - (1-\varepsilon)(\alpha+\beta) - C_\varepsilon\left(\alpha+\beta+\alpha^3+\beta^3\right) \\ &= (1-\varepsilon)\left(\|u\|_{H^1}^2 + \|v\|_{H^1}^2\right) - C_{\varepsilon,\alpha,\beta} > -\infty. \end{aligned}$$

This ends the proof. □

Next we introduce the concentration-compactness technique for Γ. Let $x \in I_e$ and $y \in I_j$, $e, j = 1, \ldots, N$, be two points of graph. We define the distance

$$d(x,y) := \begin{cases} |x-y| & \text{for} \quad e = j \\ x + y & \text{for} \quad e \neq j, \end{cases} \tag{2.10}$$

and the open ball of center x and radius r

$$B(x,r) := \{y \in \Gamma \, : \, d(x,y) < r\}.$$

Let $x \in I_e$, we set the L^p norm restricted to the ball $B(x,r)$

$$\|u\|^p_{L^p(B(x,r))} = \int_{\{y \in I_e \, : \, |x-y|<r\}} |u_e(y)|^p \, dy + \sum_{\substack{j=1 \\ j \neq e}}^{N} \int_{\{y \in I_j \, : \, x+y<r\}} |u_j(y)|^p \, dy. \tag{2.11}$$

For each minimizing sequence $\{(u_n, v_n)\} \subset X$ of $J(\alpha, \beta)$ we introduce the following sequence (Lévy concentration functions) $\rho_n : [0, \infty) \to [0, \alpha + \beta]$ by

$$\rho_n(r) = \sup_{x \in \Gamma} \left(\|u_n\|^2_{B(x,r)} + \|v_n\|^2_{B(x,r)} \right).$$

Since $\{(u_n, v_n)\}$ is a minimizing sequence, then $\{\rho_n\}$ is a uniformly bounded sequence of nondecreasing functions on $[0, \infty)$. Moreover, it has a subsequence which is still denoted by $\{\rho_n\}$ that converges pointwise and uniformly on compact sets to a nonnegative nondecreasing function $\rho(r) : [0, \infty) \to [0, \alpha + \beta]$ (see [2, Lemma 3.2]). Define

$$\tau = \lim_{r \to \infty} \rho(r) = \lim_{r \to \infty} \lim_{n \to \infty} \rho_n(r) = \lim_{r \to \infty} \lim_{n \to \infty} \sup_{x \in \Gamma} \left(\|u_n\|^2_{B(x,r)} + \|v_n\|^2_{B(x,r)} \right).$$

Since $\|u_n\|_2^2 = \alpha$ and $\|v_n\|_2^2 = \beta$, it is clear that $\tau \in [0, \alpha + \beta]$. By concentration-compactness lemma for star graphs [2, Lemma 3.3], we have three (mutually exclusive) possibilities.

(I) (*Compactness*) $\tau = \alpha + \beta$. Then, up to a subsequence, at least one of the two following cases occurs:

(I_1)(*Convergence*) There exists $(\Phi_1, \Phi_2) \in X$ such that $u_n \to \Phi_1$ and $v_n \to \Phi_2$ in $L^p(\Gamma)$ as $n \to \infty$ for all $2 \leq p \leq \infty$.

(I_2) (*Runaway*) There exists j^* such that for any $e \neq j^*$, $r > 0$, and $2 \leq p \leq \infty$

$$\|(u_n)_e\|_p \to 0, \quad \|(v_n)_e\|_p \to 0, \quad \|u_n\|_{L^p(B(0,r))} \to 0, \quad \|v_n\|_{L^p(B(0,r))} \to 0$$

as $n \to \infty$.

(II) (*Vanishing*) $\tau = 0$. Then, up to a subsequence, $u_n \to 0$ and $v_n \to 0$ in $L^p(\Gamma)$ as $n \to \infty$ for all $2 < p \leq \infty$.

(III) (*Dichotomy*) $0 < \tau < \alpha + \beta$.

In what follows we show that case (I_1) holds ruling out consequently $(II), (III)$, and (I_2). The following two propositions are used to show that vanishing case does not hold.

Proposition 2.8. *Let $\{(u_n, v_n)\} \subset X$ be a minimizing sequence for $J(\alpha, \beta)$. Then there exist constants $A > 0$ and $\lambda > 0$ such that*

(i) $\|u_n\|_{H^1} + \|v_n\|_{H^1} \leq A$ for all $n \in \mathbb{N}$;

(ii) $\|u_n\|_4^4 + \|v_n\|_4^4 \geq \lambda$ for n large enough.

Proof. (i) From (2.7)–(2.9), inequality $J(\alpha,\beta)<0$, and $\|u_n\|_2^2=\alpha$, $\|v_n\|_2^2=\beta$ we obtain

$$\begin{aligned}&\|u_n\|_{H^1}^2+\|v_n\|_{H^1}^2=E(u_n,v_n)+\gamma\left(|u_{n1}(0)|^2+|v_{n1}(0)|^2\right)+\frac{1}{2}\left(a\,\|u_n\|_4^4+c\,\|v_n\|_4^4\right)\\&+b\|u_nv_n\|_2^2+\alpha+\beta\le\sup_n E(u_n,v_n)+C\left(\|u_n'\|_2+\|v_n'\|_2\right)+\alpha+\beta\\&\le C\left(1+\|u_n\|_{H^1}+\|v_n\|_{H^1}\right).\end{aligned}$$

Since quadratic term $\|u_n\|_{H^1}^2+\|v_n\|_{H^1}^2$ is less than the linear one, the existence of the desired bound A follows.

To prove (ii), let us assume that such λ does not exist, then

$$\liminf_{n\to\infty}\left(\|u_n\|_4^4+\|v_n\|_4^4\right)=0. \tag{2.12}$$

It follows from Gagliardo–Nirenberg inequality (1.6) (for $p=4, q=\infty$) and (i) that $(|u_{n1}(0)|+|v_{n1}(0)|)\le C\left(\|u_n\|_4^{2/3}+\|v_n\|_4^{2/3}\right)$. Hence, by (2.12), we obtain

$$\liminf_{n\to\infty}\left(|u_{n1}(0)|+|v_{n1}(0)|\right)=0.$$

It is easily seen that there exists $C>0$ such that

$$0\le E(u_n,v_n)+C\left(\|u_n\|_4^4+\|v_n\|_4^4+\gamma(|u_{n1}(0)|^2+|v_{n1}(0)|^2)\right),$$

then $J(\alpha,\beta)=\lim_{n\to\infty}E(u_n,v_n)\ge 0$ which contradicts $J(\alpha,\beta)<0$. $\square$

Proposition 2.9. *Let $u,v\in H^1(\Gamma)$, $A,\lambda>0$. Assume $\|u\|_{H^1}+\|v\|_{H^1}\le A$ and $\|u\|_4^4+\|v\|_4^4\ge\lambda$, then there exists $B=B(A,\lambda)$ such that*

$$\sup_{x\in\Gamma}\left(\|u\|^4_{L^4(B(x,\frac12))}+\|v\|^4_{L^4(B(x,\frac12))}\right)\ge B.$$

Proof. Since $\|u\|_4^4+\|v\|_4^4\ge\lambda$, then we can choose $e_0\in\{1,\dots,N\}$ and $\lambda_{e_0}>0$ such that $\|u_{e_0}\|_4^4+\|v_{e_0}\|_4^4\ge\lambda_{e_0}$. Let $n\in I_{e_0}$ be a natural number and $f\in L^p(\Gamma)$, then

$$\|f\|^p_{L^p(B(n,\frac12))}=\begin{cases}\int_{n-\frac12}^{n+\frac12}|f_{e_0}(x)|^p\,dx & \text{for } n\ge1\\ \sum_{e=1}^N\int_0^{\frac12}|f_e(x)|^p\,dx & \text{for } n=0.\end{cases}$$

Hence

$$\begin{aligned}\|u_{e_0}\|_{H^1}^2+\|v_{e_0}\|_{H^1}^2&\le\sum_{n\in\mathbb{N}_0}\left\{\|u\|^2_{H^1(B(n,\frac12))}+\|v\|^2_{H^1(B(n,\frac12))}\right\}\\&\le\frac{A^2}{\|u_{e_0}\|_4^4+\|v_{e_0}\|_4^4}\left(\|u_{e_0}\|_4^4+\|v_{e_0}\|_4^4\right)\\&\le\sum_{n\in\mathbb{N}_0}\frac{A^2}{\|u_{e_0}\|_4^4+\|v_{e_0}\|_4^4}\left(\|u\|^4_{L^4(B(n,\frac12))}+\|v\|^4_{L^4(B(n,\frac12))}\right).\end{aligned}$$

Therefore, there must exist $n_0\in\mathbb{N}_0$ such that

$$\begin{aligned}&\|u\|^2_{H^1(B(n_0,\frac12))}+\|v\|^2_{H^1(B(n_0,\frac12))}\\&\le\frac{A^2}{\|u_{e_0}\|_4^4+\|v_{e_0}\|_4^4}\left(\|u\|^4_{L^4(B(n_0,\frac12))}+\|v\|^4_{L^4(B(n_0,\frac12))}\right).\end{aligned}\tag{2.13}$$

Now, from inequality (1.7) we have

$$\begin{aligned}&\|u\|^4_{L^4(B(n_0,\frac{1}{2}))}+\|v\|^4_{L^4(B(n_0,\frac{1}{2}))}\le C\left(\|u\|_{H^1(B(n_0,\frac{1}{2}))}\|u\|_2^3+\|v\|_{H^1(B(n_0,\frac{1}{2}))}\|v\|_2^3\right)\\&\le C\left(\|u\|^2_{H^1(B(n_0,\frac{1}{2}))}\|u\|_2^2+\|v\|^2_{H^1(B(n_0,\frac{1}{2}))}\|v\|_2^2\right)\le C\left(\|u\|^2_{H^1(B(n_0,\frac{1}{2}))}+\|v\|^2_{H^1(B(n_0,\frac{1}{2}))}\right)^2,\end{aligned} \tag{2.14}$$

where C does not depend on u and v. Then, combining (2.13) and (2.14), we get

$$\|u\|^2_{H^1(B(n_0,\frac{1}{2}))}+\|v\|^2_{H^1(B(n_0,\frac{1}{2}))}\ge\frac{\|u_{e_0}\|_4^4+\|v_{e_0}\|_4^4}{CA^2}.$$

Thus, from (2.13) we obtain

$$B:=\frac{\lambda_{e_0}^2}{CA^4}\le\|u\|^4_{L^4(B(n_0,\frac{1}{2}))}+\|v\|^4_{L^4(B(n_0,\frac{1}{2}))}.\quad\square$$

The following lemma rules out the vanishing case.

Lemma 2.10. *Let $\{(u_n,v_n)\}$ be a minimizing sequence for $J(\alpha,\beta)$, then $\tau>0$.*

Proof. It follows from Propositions 2.8 and 2.9 that there must exist a sequence $\{x_n\}\in\mathbb{R}^+$ and $B>0$ such that

$$\|u_n\|^4_{L^4(B(x_n,1/2))}+\|v_n\|^4_{L^4(B(x_n,1/2))}\ge B$$

for all n. Hence, using (2.11) and the Sobolev embedding of $H^1(\Gamma)$ into $L^\infty(\Gamma)$, we get

$$\begin{aligned}B&\le\|u_n\|^2_\infty\|u_n\|^2_{L^2(B(x_n,1/2))}+\|v_n\|^2_\infty\|v_n\|^2_{L^2(B(x_n,1/2))}\\&\le CA^2\left(\|u_n\|^2_{L^2(B(x_n,1/2))}+\|v_n\|^2_{L^2(B(x_n,1/2))}\right),\end{aligned}$$

where A is from Proposition 2.8. Thus,

$$\tau=\lim_{r\to\infty}\rho(r)\ge\rho(\frac{1}{2})=\lim_{n\to\infty}\rho_n(\frac{1}{2})\ge\frac{B}{CA^2}>0.\quad\square$$

To exclude the possibility of dichotomy case, we first define

$$E_1(u)=\|u'\|_2^2-\gamma|u_1(0)|^2-\frac{b^2-ac}{2(b-c)}\|u\|_4^4,\qquad E_2(u)=\|u'\|_2^2-\gamma|u_1(0)|^2-\frac{b^2-ac}{2(b-a)}\|u\|_4^4,$$

and minimization problems:

$$\begin{aligned}J_1(\alpha)&=\inf\left\{E_1(u):\ u\in H^1(\Gamma),\quad\|u\|_2^2=\alpha\right\},\\J_2(\beta)&=\inf\left\{E_2(u):\ u\in H^1(\Gamma),\quad\|u\|_2^2=\beta\right\}.\end{aligned} \tag{2.15}$$

The corresponding sets of minimizers for J_1 and J_2 are denoted by

$$\begin{aligned}\mathcal{G}_1(\alpha)&=\left\{u\in H^1(\Gamma)\ :\ J_1(\alpha)=E_1(u),\quad\|u\|_2^2=\alpha\right\},\\\mathcal{G}_2(\beta)&=\left\{u\in H^1(\Gamma)\ :\ J_2(\alpha)=E_2(u),\quad\|u\|_2^2=\beta\right\}.\end{aligned}$$

The following four technical propositions will be used to rule out the possibility of dichotomy of minimizing sequences.

Proposition 2.11. *Let $N\sqrt{\omega}-\gamma \le \frac{2\gamma}{N}$, then the following equalities hold:*
(i) $J_1(\alpha) = -\frac{1}{N^2}\left(\frac{1}{3}\left[\frac{b^2-ac}{2(b-c)}\right]^2\alpha^3 + \frac{b^2-ac}{2(b-c)}\gamma\alpha^2 + \gamma^2\alpha\right)$;
(ii) $J_2(\beta) = -\frac{1}{N^2}\left(\frac{1}{3}\left[\frac{b^2-ac}{2(b-a)}\right]^2\beta^3 + \frac{b^2-ac}{2(b-a)}\gamma\beta^2 + \gamma^2\beta\right)$;
(iii) $J(\alpha,\beta) = J_1(\alpha) + J_2(\beta)$.

Proof. *(i)* Since $\gamma > 0$, $\omega > \frac{\gamma^2}{N^2}$, and $N\sqrt{\omega}-\gamma \le \frac{2\gamma}{N}$, by [2, Theorem 2] (see also Remark 2.3), we have

$$\mathcal{G}_1(\alpha) = \left\{ e^{i\theta_1}\sqrt{\frac{b-c}{b^2-ac}}\,\Phi_{\omega,\gamma}(x) \,:\, \theta_1 \in \mathbb{R}\right\}$$

and

$$\mathcal{G}_2(\beta) = \left\{ e^{i\theta_2}\sqrt{\frac{b-a}{b^2-ac}}\,\Phi_{\omega,\gamma}(x) \,:\, \theta_2 \in \mathbb{R}\right\},$$

where, for $e = 1,\dots,N$, $(\Phi_{\omega,\gamma})_e = \phi_{\omega,\gamma}$, with

$$\phi_{\omega,\gamma}(x) = \sqrt{2\omega}\,\mathrm{sech}\left(\sqrt{\omega}x + \tanh^{-1}\left(\frac{\gamma}{N\sqrt{\omega}}\right)\right).$$

Therefore, $J_1(\alpha) = E_1(\sqrt{\frac{b-c}{b^2-ac}}\Phi_{\omega,\gamma})$ and $J_2(\beta) = E_2(\sqrt{\frac{b-a}{b^2-ac}}\Phi_{\omega,\gamma})$. Hence

$$\begin{aligned}
J_1(\alpha) &= N\frac{b-c}{b^2-ac}\left[\int_{\mathbb{R}+}\left|\phi'_{\omega,\gamma}(x)\right|^2dx - \frac{1}{2}\int_{\mathbb{R}+}|\phi_{\omega,\gamma}(x)|^4dx - \frac{\gamma}{N}|\phi_{\omega,\gamma}(0)|^2\right]\\
&= 2N\frac{b-c}{b^2-ac}\left[\omega^2\int_{\mathbb{R}+}\mathrm{sech}^2\left(\sqrt{\omega}x + \tanh^{-1}\left(\frac{\gamma}{N\sqrt{\omega}}\right)\right)dx\right.\\
&\left.- 2\omega^2\int_{\mathbb{R}+}\mathrm{sech}^4\left(\sqrt{\omega}x + \tanh^{-1}\left(\frac{\gamma}{N\sqrt{\omega}}\right)\right)dx - \frac{\omega\gamma}{N} + \frac{\gamma^3}{N^3}\right]\\
&= 2N\frac{b-c}{b^2-ac}\left[\frac{\omega^2}{\sqrt{\omega}}(1-\frac{\gamma}{N\sqrt{\omega}}) - \frac{2\omega^2}{\sqrt{\omega}}(1-\frac{\gamma}{N\sqrt{\omega}}) + \frac{2\omega^2}{3\sqrt{\omega}}(1-\frac{\gamma^3}{N^3\omega^{3/2}}) - \frac{\omega\gamma}{N} + \frac{\gamma^3}{N^3}\right]\\
&= -\frac{2}{3}\frac{b-c}{b^2-ac}\left(N\omega^{3/2} - \frac{\gamma^3}{N^2}\right).
\end{aligned} \tag{2.16}$$

Recalling that $\alpha = 2\frac{b-c}{b^2-ac}(N\sqrt{\omega}-\gamma)$, we get

$$\begin{aligned}
J_1(\alpha) &= -\frac{2}{3}\frac{b-c}{b^2-ac}\left(N\omega^{3/2} - \frac{\gamma^3}{N^2}\right)\\
&= -\frac{1}{N^2}\left(\frac{2}{3}\frac{b-c}{b^2-ac}\left[(N\sqrt{\omega}-\gamma)^3 + 3\Big(\gamma(N\sqrt{\omega}-\gamma)^2 + \gamma^2(N\sqrt{\omega}-\gamma)\Big)\right]\right)\\
&= -\frac{1}{N^2}\left(\frac{1}{3}\left[\frac{b^2-ac}{2(b-c)}\right]^2\alpha^3 + \frac{b^2-ac}{2(b-c)}\gamma\alpha^2 + \gamma^2\alpha\right).
\end{aligned}$$

In the same manner we can show *(ii)*.

(iii) Next we prove $J(\alpha,\beta) \ge J_1(\alpha) + J_2(\beta)$. Using the Cauchy–Schwarz inequality and the Young inequality, we have

$$\int_\Gamma |u|^2|v|^2dx \le \sqrt{\frac{b-a}{b-c}}\|u\|_4^2\sqrt{\frac{b-c}{b-a}}\|v\|_4^2 \le \frac{1}{2}\left(\frac{b-a}{b-c}\|u\|_4^4 + \frac{b-c}{b-a}\|u\|_4^4\right).$$

Hence

$$\begin{aligned} E(u,v) &\geq \|u'\|_2^2 + \|v'\|_2^2 - \gamma\left(|u_1(0)|^2 + |v_1(0)|^2\right) \\ &- \frac{1}{2}\left(a\|u\|_4^4 + c\|v\|_4^4\right) - \frac{b}{2}\left(\frac{b-a}{b-c}\|u\|_4^4 + \frac{b-c}{b-a}\|v\|_4^4\right) \\ &= \|u'\|_2^2 + \|v'\|_2^2 - \gamma\left(|u_1(0)|^2 + |v_1(0)|^2\right) - \frac{1}{2}\left(\frac{b^2-ac}{b-c}\|u\|_4^4 + \frac{b^2-ac}{b-a}\|v\|_4^4\right) \\ &= E_1(u) + E_2(v). \end{aligned} \tag{2.17}$$

Taking the infimum on u and v, we obtain $J(\alpha,\beta) \geq J_1(\alpha) + J_2(\beta)$. On the other hand, observe that $\left\|\sqrt{\frac{b-c}{b^2-ac}}\Phi_{\omega,\gamma}\right\|_2^2 = \alpha$ and $\left\|\sqrt{\frac{b-a}{b^2-ac}}\Phi_{\omega,\gamma}\right\|_2^2 = \beta$ for any $\omega > 0$ fixed. Thus, we have

$$\begin{aligned} J(\alpha,\beta) &\leq E\left(\sqrt{\frac{b-c}{b^2-ac}}\Phi_{\omega,\gamma}, \sqrt{\frac{b-a}{b^2-ac}}\Phi_{\omega,\gamma}\right) \\ &= E_1\left(\sqrt{\frac{b-c}{b^2-ac}}\Phi_{\omega,\gamma}\right) + E_2\left(\sqrt{\frac{b-a}{b^2-ac}}\Phi_{\omega,\gamma}\right) \\ &= J_1(\alpha) + J_2(\beta) \leq J(\alpha,\beta). \end{aligned}$$

Hence we arrive at (iii), and the result is proved. □

Proposition 2.12. *Let $\delta_1 \in (0,\alpha)$ and $\delta_2 \in (0,\beta)$, then*

$$J_1(\alpha) < J_1(\delta_1) + J_1(\alpha-\delta_1) \quad \textit{and} \quad J_2(\beta) < J_2(\delta_2) + J_2(\beta-\delta_2).$$

Proof. We claim that if $\eta > 1$ and $\alpha > 0$, then $J_1(\eta\alpha) < \eta J_1(\alpha)$. To prove this inequality, consider a minimizing sequence $\{u_n\}$ for $J_1(\alpha)$, and set $\widetilde{u}_n = \sqrt{\eta}u_n$ for all n. Then $\|\widetilde{u}_n\|_2^2 = \eta\alpha$ and hence

$$\begin{aligned} J_1(\eta\alpha) &\leq E_1(\widetilde{u}_n) = \eta\|u_n'\|_2^2 - \gamma\eta|u_{n1}(0)|^2 - \eta^2\frac{b^2-ac}{2(b-c)}\|u_n\|_4^4 \\ &= \eta E_1(u_n) - \left(\eta^2-\eta\right)\frac{b^2-ac}{2(b-c)}\|u_n\|_4^4. \end{aligned} \tag{2.18}$$

On the other hand, repeating the proof of Proposition 2.8, we can show that there exists a constant $\lambda > 0$ such that $\|u_n\|_4^4 \geq \lambda$ for n large enough. Hence, taking $n \to \infty$ in (2.18), we obtain

$$J_1(\eta\alpha) \leq \eta J_1(\alpha) - \left(\eta^2-\eta\right)\frac{b^2-ac}{2(b-c)}\lambda < \eta J_1(\alpha).$$

Without loss of generality, we suppose that $\delta_1 > \alpha - \delta_1$. Then, by the claim proved above, we have

$$\begin{aligned} J_1(\alpha) &= J_1\left(\delta_1(1+\frac{\alpha-\delta_1}{\delta_1})\right) < \left(1+\frac{\alpha-\delta_1}{\delta_1}\right)J_1(\delta_1) \\ &< J_1(\delta_1) + \frac{\alpha-\delta_1}{\delta_1}\left(\frac{\delta_1}{\alpha-\delta_1}J_1(\alpha-\delta_1)\right) = J_1(\delta_1) + J_1(\alpha-\delta_1). \end{aligned}$$

Also, if $\delta_1 = \alpha - \delta_1$, then we get

$$J_1(\alpha) = J_1(2\delta_1) < 2J_1(\delta_1) = J_1(\delta_1) + J_1(\alpha-\delta_1).$$

The result for J_2 can be derived in an analogous way. □

The following proposition states strict subadditivity of J.

Proposition 2.13. *Let $\delta_1 \in [0,\alpha]$ and $\delta_2 \in [0,\beta]$ be such that $0<\delta_1+\delta_2<\alpha+\beta$. Then*

$$J(\alpha,\beta) < J(\delta_1,\delta_2) + J(\alpha-\delta_1,\beta-\delta_2).$$

Proof. We divide the proof into the following three cases:

Case 1. Let $\delta_1 \in (0,\alpha)$ and $\delta_2 \in (0,\beta)$. Using Proposition 2.11(iii) and Proposition 2.12, we obtain

$$J(\alpha,\beta) = J_1(\alpha) + J_2(\beta) < J_1(\delta_1) + J_1(\alpha-\delta_1) + J_2(\delta_2) + J_2(\alpha-\delta_2). \tag{2.19}$$

On the other hand, from (2.17) we get

$$J_1(\delta_1) + J_2(\delta_2) \le J(\delta_1,\delta_2) \quad \text{and} \quad J_1(\alpha-\delta_1) + J_2(\beta-\delta_2) \le J(\alpha-\delta_1,\beta-\delta_2). \tag{2.20}$$

Thus, combining (2.19) and (2.20), we obtain

$$J(\alpha,\beta) < J(\delta_1,\delta_2) + J(\alpha-\delta_1,\beta-\delta_2),$$

as desired.

Case 2. Assume that $\delta_1 = 0$ and $\delta_2 \in (0,\beta]$. We consider the following variational problem

$$J(0,\delta_2) = \inf\left\{\|v'\|_2^2 - \gamma|v_1(0)|^2 - \frac{c}{2}\|v\|_4^4 \,:\, v \in H^1(\Gamma),\ \|v\|_2^2 = \delta_2\right\}. \tag{2.21}$$

For $c>0$ the minimizer of (2.21) is given by $\Phi_2(x) = (\phi_2(x))_{e=1}^N$, where

$$\phi_2(x) = \sqrt{\frac{2\omega_2}{c}}\operatorname{sech}\left(\sqrt{\omega_2}x + \tanh^{-1}\left(\frac{\gamma}{N\sqrt{\omega_2}}\right)\right)$$

and $\omega_2 = \left(\frac{c\delta_2+2\gamma}{2N}\right)^2$ (see formulas (5.4) and (5.1) in [2]). Therefore,

$$J(0,\delta_2) = N\|\phi_2'\|_2^2 - \gamma\,|\phi_2(0)|^2 - N\frac{c}{2}\|\phi_2\|_4^4 = -\frac{1}{N^2}\left(\frac{c^2}{12}\delta_2^3 + \frac{c}{2}\gamma\delta_2^2 + \gamma^2\delta_2\right).$$

Now, from Proposition 2.11(ii), we have

$$J_2(\delta_2) = -\frac{1}{N^2}\left(\frac{1}{3}\left[\frac{b^2-ac}{2(b-a)}\right]^2\delta_2^3 + \frac{b^2-ac}{2(b-a)}\gamma\delta_2^2 + \gamma^2\delta_2\right).$$

Hence $J(0,\delta_2) > J_2(\delta_2)$. Then

$$J(\alpha,\beta) = J_1(\alpha) + J_2(\beta) \le J_1(\alpha) + J_2(\delta_2) + J_2(\beta-\delta_2) < J(0,\delta_2) + J(\alpha,\beta-\delta_2).$$

Case 3. Assume that $\delta_1 \in (0,\alpha]$ and $\delta_2 = 0$. Analogously to Case 2, we consider the following variational problem

$$J(\delta_1,0) = \inf\left\{\|u'\|_2^2 - \gamma|u_1(0)|^2 - \frac{a}{2}\|u\|_4^4 \,:\, u \in H^1(\Gamma),\ \|u\|_2^2 = \delta_1\right\}.$$

Since $a>0$, the minimizer of (2.21) is given by $\Phi_1(x) = (\phi_1(x))_{e=1}^N$, where

$$\phi_1(x) = \sqrt{\frac{2\omega_1}{a}}\operatorname{sech}\left(\sqrt{\omega_1}x + \tanh^{-1}\left(\frac{\gamma}{N\sqrt{\omega_1}}\right)\right),$$

where $\omega_1 = \left(\frac{a\delta_1+2\gamma}{2N}\right)^2$. Therefore,

$$J(\delta_1, 0) = -\frac{1}{N^2}\left(\frac{a^2}{12}\delta_1^3 + \frac{a}{2}\gamma\delta_1^2 + \gamma^2\delta_1\right). \tag{2.22}$$

Now, from Proposition 2.11(i) we have

$$J_1(\delta_1) = -\frac{1}{N^2}\left(\frac{1}{3}\left[\frac{b^2-ac}{2(b-c)}\right]^2\delta_1^3 + \frac{b^2-ac}{2(b-c)}\gamma\delta_1^2 + \gamma^2\delta_1\right).$$

Hence from (2.22) we get that $J(\delta_1, 0) > J_1(\delta_1)$. Then

$$J(\alpha, \beta) = J_1(\alpha) + J_2(\beta) \le J_1(\delta_1) + J_1(\alpha - \delta_1) + J_2(\beta) < J(\delta_1, 0) + J(\alpha - \delta_1, \beta). \quad \square$$

We introduce two sequences $\{(f_n, g_n)\}$ and $\{(h_n, l_n)\}$ associated with an arbitrary minimizing sequence $\{(u_n, v_n)\}$ in the following way.

Let $\sigma, \kappa \in C_0^\infty(\mathbb{R}^+)$. We assume that σ is supported on $[0, 2]$, $\sigma \equiv 1$ on $[0, 1]$ and $\sigma^2 + \kappa^2 = 1$ on $\mathbb{R}^+$. Set $\sigma_r(x) = \sigma(\frac{x}{r})$ and $\kappa_r(x) = \kappa(\frac{x}{r})$ for $r > 0$. Let $\varepsilon > 0$, for sufficiently large r we have $\tau - \varepsilon < \rho(r) \le \rho(2r) \le \tau$. We can choose N large enough so that

$$\tau - \varepsilon < \rho_n(r) \le \rho_n(2r) < \tau + \varepsilon$$

for all $n \ge N$. Consequently, for each $n \ge N$ we can find $x_n \in \Gamma$ such that

$$\begin{aligned}\left(\|u_n\|^2_{L^2(B(x_n,r))} + \|v_n\|^2_{L^2(B(x_n,r))}\right) &> \tau - \varepsilon,\\ \left(\|u_n\|^2_{L^2(B(x_n,2r))} + \|v_n\|^2_{L^2(B(x_n,2r))}\right) &< \tau + \varepsilon.\end{aligned} \tag{2.23}$$

Let $x \in \Gamma$, $d(x, x_n)$ denotes the distance between x and x_n given by (2.10). Define (f_n, g_n) and (h_n, l_n) such that $f_{n,e}(x) = \sigma_r(d(x, x_n))u_{n,e}(x)$, $g_{n,e}(x) = \sigma_r(d(x, x_n))v_{n,e}(x)$, $h_{n,e}(x) = \kappa_r(d(x, x_n))u_{n,e}(x)$ and $l_{n,e}(x) = \kappa_r(d(x, x_n))v_{n,e}(x)$ for $e = 1, \ldots, N$. Observe that $Q(f_n) \ge \|u_n\|^2_{L^2(B(x_n,r))}$ and $Q(g_n) \ge \|v_n\|^2_{L^2(B(x_n,r))}$. Since $\sigma \le 1$ and $\operatorname{supp}\sigma_r \subseteq \{x \in \Gamma : d(x, x_n) \le 2r\}$, we have $Q(f_n) \le \|u_n\|^2_{L^2(B(x_n,2r))}$ and $Q(g_n) \le \|v_n\|^2_{L^2(B(x_n,2r))}$.

Observe that $\alpha - \|u_n\|^2_{L^2(B(x_n,2r))} \le Q(h_n)$ and $\beta - \|v_n\|^2_{L^2(B(x_n,2r))} \le Q(l_n)$. Since $\kappa \le 1$ and $\{x \in \Gamma : d(x, x_n) > 2r\} \subseteq \operatorname{supp}\kappa_r$, we have $Q(h_n) \le \alpha - \|u_n\|^2_{L^2(B(x_n,r))}$ and $Q(l_n) \le \beta - \|v_n\|^2_{L^2(B(x_n,r))}$.

Proposition 2.14. *Let $\{(u_n, v_n)\}$ be a minimizing sequence for (2.2) and sequences $\{(f_n, g_n)\}$, $\{(h_n, l_n)\}$ be defined above. Then for every $\varepsilon > 0$ there exists $N > 0$ such that for $n \ge N$:*
(i) $|Q(f_n) + Q(g_n) - \tau| < \varepsilon$,
(ii) $|Q(h_n) + Q(l_n) - (\alpha + \beta - \tau)| < \varepsilon$,
(iii) $E(f_n, g_n) + E(h_n, l_n) \le E(u_n, v_n) + C\varepsilon$ for some $C > 0$ independent of n.

Proof. The proof of $(i), (ii)$ is a direct consequence of (2.23). To prove (iii) notice that for $\frac{1}{r} < \varepsilon$

$$\begin{aligned}E(f_n, g_n) = &\int_\Gamma \sigma_r^2\left[(u_n')^2 + (v_n')^2 - \frac{a}{2}|u_n|^4 - \frac{c}{2}|u_n|^4 - b|u_n|^2|v_n|^2\right]dx\\ &- \gamma|\sigma_r(x_n)|^2\left(|u_{n1}(0)|^2 + |v_{n1}(0)|^2\right) + \int_\Gamma(\sigma_r^2 - \sigma_r^4)\left[\frac{a}{2}|u_n|^4 + \frac{c}{2}|v_n|^4 + b|u_n|^2|v_n|^2\right]dx\\ &+ \int_\Gamma(\sigma_r')^2\Big[|u_n|^2 + |v_n|^2 + 2\sigma_r'\sigma_r\left(u_n'u_n + v_n'v_n\right)\Big]dx\\ &\le \int_\Gamma \sigma_r^2\left[(u_n')^2 + (v_n')^2 - \frac{a}{2}|u_n|^4 - \frac{c}{2}|u_n|^4 - b|u_n|^2|v_n|^2\right]dx\\ &- \gamma|\sigma_r(x_n)|^2\left(|u_{n1}(0)|^2 + |v_{n1}(0)|^2\right) + C\varepsilon.\end{aligned}$$

Indeed, observe that $|\sigma_r'|_\infty = |\sigma'|_\infty / r \le C\varepsilon$ (since $1/r < \varepsilon$). Under the integral σ_r has to be read as $(\sigma_r)_{e=1}^N$. Moreover, introducing

$$\left(\chi_{(B(x_n,2r)\setminus B(x_n,r))}\right)_e(x) = \begin{cases} 1 & \text{if } r \le d(x,x_n) < 2r \\ 0 & \text{otherwise,} \end{cases}$$

and using (2.23), we get

$$\int_\Gamma \left(\sigma_r^2 - \sigma_r^4\right) |u_n|^4 \, dx \le \|u_n\|_\infty^2 \sum_{e=1}^N \int_{I_e} \left(\chi_{(B(x_n,2r)\setminus B(x_n,r))}\right)_e(x) \, |u_{n,e}(x)|^2 \, dx \le C\varepsilon,$$

and

$$\begin{aligned} &\int_\Gamma \left(\sigma_r^2 - \sigma_r^4\right) |u_n|^2 |v_n|^2 \, dx \\ &\le \|u_n\|_\infty^2 \sum_{e=1}^N \int_{I_e} \left(\chi_{(B(x_n,2r)\setminus B(x_n,r))}\right)_e(x) \, |v_{n,e}(x)|^2 \, dx \le C\varepsilon, \end{aligned}$$

where C denotes constant independent of r and n. Similarly, we obtain

$$\begin{aligned} E(h_n, l_n) \le \int_\Gamma \kappa_r^2 \left[(u_n')^2 + (v_n')^2 - \frac{a}{2} |u_n|^4 - \frac{c}{2} |u_n|^4 - b |u_n|^2 |v_n|^2 \right] dx \\ - \gamma |\kappa_r(x_n)|^2 \left(|u_{n1}(0)|^2 + |v_{n1}(0)|^2 \right) + C\varepsilon. \end{aligned}$$

Thus, since $\sigma^2 + \kappa^2 \equiv 1$ on $\mathbb{R}^+$, we get (iii). □

Below we rule out the dichotomy of the minimizing sequences.

Lemma 2.15. *Let $\{(u_n, v_n)\}$ be a minimizing sequence for $J(\alpha, \beta)$. Then the case of dichotomy cannot occur.*

Proof. Assume that $\tau \in (0, \alpha + \beta)$. Let $\{(f_n, g_n)\}$ and $\{(h_n, l_n)\}$ be the sequences from Proposition 2.14, then, up to subsequences (using the fact that minimizing sequence is L^2-bounded), we have

$$\begin{gathered} |Q(f_n) - \alpha + \delta_1| < \frac{1}{2n}, \; |Q(g_n) - \beta + \delta_2| < \frac{1}{2n}, \; |Q(h_n) - \delta_1| < \frac{1}{2n}, \; |Q(l_n) - \delta_2| < \frac{1}{2n}, \\ E(f_n, g_n) + E(h_n, l_n) \le E(u_n, v_n) + \frac{1}{n}, \end{gathered}$$

where $\delta_1 \in [0, \alpha]$ and $\delta_2 \in [0, \beta]$ and, by Proposition 2.14, $\tau = \alpha + \beta - (\delta_1 + \delta_2)$. For $\delta_1 > 0, \delta_2 > 0, \alpha - \delta_1 > 0, \beta - \delta_2 > 0$ we define

$$a_{1n} = \sqrt{\frac{\alpha - \delta_1}{Q(f_n)}}, \quad a_{2n} = \sqrt{\frac{\beta - \delta_2}{Q(g_n)}}, \quad b_{1n} = \sqrt{\frac{\delta_1}{Q(h_n)}}, \quad b_{2n} = \sqrt{\frac{\delta_2}{Q(l_n)}}.$$

Thus, we have

$$\begin{aligned} &a_{1n}, a_{2n}, b_{1n}, b_{2n} \underset{n\to\infty}{\to} 1, \\ &Q(a_{1n} f_n) = \alpha - \delta_1, \; Q(a_{2n} g_n) = \beta - \delta_2, \; Q(b_{1n} h_n) = \delta_1, \; Q(b_{2n} l_n) = \delta_2. \end{aligned}$$

Consequently

$$\begin{aligned} J(\alpha - \delta_1, \beta - \delta_2) \le E(a_{1n} f_n, a_{2n} g_n) \le E(f_n, g_n) + o(1), \\ J(\delta_1, \delta_2) \le E(b_{1n} h_n, b_{2n} l_n) \le E(h_n, l_n) + o(1). \end{aligned} \tag{2.24}$$

By Proposition 2.14(*iii*) and (2.24), we get

$$J(\alpha-\delta_1,\beta-\delta_2)+J(\delta_1,\delta_2)\le E(u_n,v_n)+o(1),$$

which implies

$$J(\alpha-\delta_1,\beta-\delta_2)+J(\delta_1,\delta_2)\le J(\alpha,\beta).$$

This contradicts Proposition 2.13.

Now assume that one of the numbers $\delta_1,\delta_2,\alpha-\delta_1,\beta-\delta_2$ is zero. Without loss of generality, we can assume $\delta_1=0$, then $Q(h_n)\to 0$ and $\delta_2>0$ due to $\tau<\alpha+\beta$. Therefore, by the Gagliardo–Nirenberg inequality,

$$\begin{aligned}E(h_n,l_n)+o(1)&\ge\left(\|h_n'\|_2^2+\|l_n'\|_2^2-\gamma\,|l_{n1}(0)|^2-\frac{c}{2}\,\|l_n\|_4^4\right)+o(1)\\&\ge\left(\|l_n'\|_2^2-\gamma\,|l_{n1}(0)|^2-\frac{c}{2}\,\|l_n\|_4^4\right)+o(1)\ge J(0,\delta_2)+o(1).\end{aligned}$$

Combining this inequality with the first one from (2.24), we arrive at the contradiction again. □

Since we eliminated the vanishing and the dichotomy cases, it follows from the concentration-compactness lemma [2] that $\tau=\alpha+\beta$, that is, we have compactness case. It only remains to prove that the minimizing sequence is not *runaway*.

Lemma 2.16. *Let $\{(u_n,v_n)\}$ be a minimizing sequence for $J(\alpha,\beta)$. Then, up to subsequence, it converges in X to some (Φ_1,Φ_2), which is a minimizer for $J(\alpha,\beta)$, that is, $\|\Phi_1\|_2^2=\alpha$, $\|\Phi_2\|_2^2=\beta$, and $E(\Phi_1,\Phi_2)=J(\alpha,\beta)$.*

Proof. Set

$$\begin{aligned}E_0(u,v)&=\|u'\|_2^2+\|v'\|_2^2-\frac{1}{2}\left(a\,\|u\|_4^4+c\,\|v\|_4^4\right)-b\|uv\|_2^2,\\J_0(\alpha,\beta)&=\inf\left\{E_0(u,v)\,:\,u,v\in H^1(\Gamma),\ \|u\|_2^2=\alpha,\ \|v\|_2^2=\beta\right\}.\end{aligned}$$

By absurd, suppose that $\{(u_n,v_n)\}$ is the *runaway* sequence. Then from Proposition 2.8(*i*) and the Gagliardo–Nirenberg inequality, we have that $|u_{n,e}(0)|\to 0$ and $|v_{n,e}(0)|\to 0$, for $e\ne j^*$ (j^* is from definition (I_2) of runaway sequence) which implies that $\lim_{n\to\infty}(E(u_n,v_n)-E_0(u_n,v_n))=0$. Hence,

$$J_0(\alpha,\beta)\le J(\alpha,\beta). \tag{2.25}$$

Take $u,v\in H^1(\Gamma)$ such that $\|u\|_2^2=\alpha$ and $\|v\|_2^2=\beta$, and let $u^*,v^*\in H^1(\Gamma)$ be their symmetric rearrangements. Then, by Proposition A.17 in the Appendix, we obtain $\|u^*\|_2^2=\alpha$, $\|v^*\|_2^2=\beta$, and $E_0(u,v)\ge E_0^*(u^*,v^*)$, where

$$E_0^*(u^*,v^*)=\frac{4}{N^2}\left(\|(u^*)'\|_2^2+\|(v^*)'\|_2^2\right)-\frac{1}{2}\left(a\,\|u^*\|_4^4+c\,\|v^*\|_4^4\right)-b\|u^*v^*\|_2^2.$$

Since rearrangements maintain the mass constraint, the last inequality implies

$$J_0(\alpha,\beta)\ge\inf\left\{E_0^*(u,v):\ u,v\in H_s^1(\Gamma),\ \|u\|_2^2=\alpha,\ \|v\|_2^2=\beta\right\}, \tag{2.26}$$

where $H_s^1(\Gamma)=\{u\in H^1(\Gamma):u_1(x)=\cdots=u_N(x),x>0\}$. It is easily seen that right-hand side of (2.26) reduces to N copies of the following problem on $\mathbb{R}^+$

$$\inf\left\{E_{\mathbb{R}^+}^*(\psi,\varphi):\ \psi,\varphi\in H^1(\mathbb{R}^+),\ \|\psi\|_2^2=\frac{\alpha}{N},\ \|\varphi\|_2^2=\frac{\beta}{N}\right\},$$

where
$$E^*_{\mathbb{R}^+}(\psi,\varphi)=\frac{4}{N^2}\left(\|\psi'\|_2^2+\|\varphi'\|_2^2\right)-\frac{1}{2}\left(a\|\psi\|_4^4+c\|\varphi\|_4^4\right)-b\|\psi\varphi\|_2^2.$$
Set for $\lambda>0$ the rescaling $\psi_\lambda(x)=\sqrt{\lambda}\psi(\lambda x)$ and $\varphi_\lambda(x)=\sqrt{\lambda}\varphi(\lambda x)$, then
$$E^*_{\mathbb{R}^+}(\psi_\lambda,\varphi_\lambda)=\frac{4\lambda^2}{N^2}\left(\|\psi'\|_2^2+\|\varphi'\|_2^2\right)-\frac{\lambda}{2}\left(a\|\psi\|_4^4+c\|\varphi\|_4^4\right)-b\lambda\|\psi\varphi\|_2^2.$$
Choosing $\lambda=\frac{N^2}{4}$, we get
$$J_0(\alpha,\beta)\geq\frac{N^3}{4}d_{\mathbb{R}^+},$$
where
$$d_{\mathbb{R}^+}=\inf\left\{E_{\mathbb{R}^+}(\psi,\varphi):\ \psi,\varphi\in H^1(\mathbb{R}^+),\ \|\psi\|_2^2=\frac{\alpha}{N},\ \|\varphi\|_2^2=\frac{\beta}{N}\right\},\tag{2.27}$$
$$E_{\mathbb{R}^+}(\psi,\varphi)=\|\psi'\|_2^2+\|\varphi'\|_2^2-\frac{1}{2}\left(a\|\psi\|_4^4+c\|\varphi\|_4^4\right)dx-b\|\psi\varphi\|_2^2.$$
By [41, Theorem 2.1], the solution to minimization problem (2.27) on $\mathbb{R}^+$ is $(\psi_{\tilde\omega},\varphi_{\tilde\omega})=\left(\sqrt{\frac{b-c}{b^2-ac}}\phi_{\tilde\omega},\sqrt{\frac{b-a}{b^2-ac}}\phi_{\tilde\omega}\right)$, where $\phi_{\tilde\omega}(x)=\sqrt{2\tilde\omega}\,\mathrm{sech}(\sqrt{\tilde\omega}x)$ and $\tilde\omega$ is such that $\|\psi_{\tilde\omega}\|_2^2=\frac{\alpha}{N}$ and $\|\varphi_{\tilde\omega}\|_2^2=\frac{\beta}{N}$. Then we get
$$J_0(\alpha,\beta)\geq\frac{N^3}{4}E_{\mathbb{R}^+}\left(\psi_{\tilde\omega},\varphi_{\tilde\omega}\right)=-\frac{2}{3}\left(\frac{N^3}{4}\tilde\omega^{3/2}\right)\frac{2b-a-c}{b^2-ac}.\tag{2.28}$$
By formula (3.3) in [41], we have $\frac{\alpha}{N}=2\sqrt{\tilde\omega}\frac{b-c}{b^2-ac}$. Recalling that $\alpha=2\frac{b-c}{b^2-ac}(N\sqrt{\omega}-\gamma)$, we obtain $N\sqrt{\tilde\omega}=N\sqrt{\omega}-\gamma$. Then from (2.28) we get
$$J_0(\alpha,\beta)\geq-\frac{2}{3}\left(\frac{N^2}{4}\tilde\omega\left(N\sqrt{\omega}-\gamma\right)\right)\frac{2b-a-c}{b^2-ac}.$$

By Proposition 2.11(iii) and formula (2.16), we conclude
$$\begin{aligned}J(\alpha,\beta)&=-\frac{2}{3}\left(N\omega^{3/2}-\frac{\gamma^3}{N^2}\right)\frac{2b-a-c}{b^2-ac}\\&=-\frac{2}{3}\left(\omega\left(N\sqrt{\omega}-\gamma\right)+\gamma\left(\omega-\frac{\gamma^2}{N^2}\right)\right)\frac{2b-a-c}{b^2-ac}.\end{aligned}\tag{2.29}$$
Since $N\sqrt{\tilde\omega}=(N\sqrt{\omega}-\gamma)\leq\frac{2\gamma}{N}$, then $\frac{N^2}{4}\tilde\omega\leq\frac{\gamma^2}{N^2}<\omega$. Hence, by (2.28) and (2.29), we deduce that $J_0(\alpha,\beta)>J(\alpha,\beta)$ which contradicts (2.25). Therefore, $\{(u_n,v_n)\}$ is not *runaway*, and it converges, up to subsequence, to (Φ_1,Φ_2) in $L^p(\Gamma)\times L^p(\Gamma)$ for $p\geq 2$ and weakly in $H^1(\Gamma)\times H^1(\Gamma)$. In particular, $Q(\Phi_1)=\alpha$ and $Q(\Phi_2)=\beta$, and, by the weak lower semicontinuity of the H^1 norm, we have
$$E(\Phi_1,\Phi_2)\leq\lim_{n\to\infty}E(u_n,v_n)=J(\alpha,\beta),$$
whence $E(\Phi_1,\Phi_2)=J(\alpha,\beta)$ and $(\Phi_1,\Phi_2)\in\mathcal{G}(\alpha,\beta)$. Using $E(\Phi_1,\Phi_2)=\lim_{n\to\infty}E(u_n,v_n)$ and $\|u_n-\Phi_1\|_p\to 0$, $\|v_n-\Phi_2\|_p\to 0$, $p\geq 2$, and the weak convergence, we conclude
$$\|(\Phi_1,\Phi_2)\|_X=\lim_{n\to\infty}\|(u_n,v_n)\|_X.$$
Indeed, it is sufficient to observe that
$$\|(u_n,v_n)\|_X^2=E(u_n,v_n)+\frac{1}{2}\left(a\|u_n\|_4^4+b\|v_n\|_4^4\right)+b\|u_nv_n\|_2^2+\gamma(|u_{n1}(0)|^2+|v_{n1}(0)|^2).$$
Finally, since X is the Hilbert space, we conclude that $(u_n,v_n)\to(\Phi_1,\Phi_2)$ in X. □

Remark 2.17. In [41] the authors consider minimizing problem on the line

$$d_{\mathbb{R}} = \inf\{E_{\mathbb{R}} : (\psi, \varphi) \in H^1(\mathbb{R}),\ \|\psi\|_2^2 = \tilde{\alpha},\ \|\varphi\|_2^2 = \tilde{\beta}\},$$

where $E_{\mathbb{R}} = E_{\mathbb{R}+}$, and $\tilde{\alpha} = 4\sqrt{\tilde{\omega}}\frac{b-c}{b^2-ac}$, $\tilde{\beta} = 4\sqrt{\tilde{\omega}}\frac{b-a}{b^2-ac}$. Assuming $\tilde{\alpha} = \frac{2\alpha}{N}, \tilde{\beta} = \frac{2\beta}{N}$, recalling that $d_{\mathbb{R}} = E_{\mathbb{R}}\left(\sqrt{\frac{b-c}{b^2-ac}}\phi_{\tilde{\omega}}, \sqrt{\frac{b-a}{b^2-ac}}\phi_{\tilde{\omega}}\right)$, and taking into account the fact that $\phi_{\tilde{\omega}}(x)$ is an even function on the line, we get $d_{\mathbb{R}} = 2d_{\mathbb{R}+}$.

Proof of Theorem 2.2. (i) The statement follows immediately from Lemma 2.16.

(ii) We argue by a contradiction. Suppose that (2.4) is false. Then, there exist $\varepsilon > 0$ and a subsequence $\{(u_{n_k}, v_{n_k})\}$ such that

$$\inf_{(\Phi_1,\Phi_2)\in\mathcal{G}(\alpha,\beta)} \left\|(u_{n_k}, v_{n_k}) - (\Phi_1, \Phi_2)\right\|_X \geq \varepsilon \text{ for any } k \in \mathbb{N}.$$

But $\{(u_{n_k}, v_{n_k})\}$ is also a minimizing sequence for $J(\alpha,\beta)$, hence there exists $(\tilde{\Phi}_1, \tilde{\Phi}_2) \in \mathcal{G}(\alpha,\beta)$ such that

$$\liminf_{k\to\infty} \left\|(u_{n_k}, v_{n_k}) - (\tilde{\Phi}_1, \tilde{\Phi}_2)\right\|_X = 0,$$

which gives a contradiction. $\square$

2.3. Orbital stability of ground states

This subsection is devoted to the proof of Theorem 2.4 and Corollary 2.6. First, as an easy consequence of Theorem 2.2, we conclude that the set of minimizers $\mathcal{G}(\alpha,\beta)$ is stable under the flow generated by system (1.2).

Corollary 2.18. *Let $\varepsilon > 0$. Then there exists $\eta > 0$ with the following property: if $(u_0, v_0) \in X$ satisfies*

$$\inf_{(\Phi_1,\Phi_2)\in\mathcal{G}(\alpha,\beta)} \|(u_0, v_0) - (\Phi_1, \Phi_2)\|_X < \eta,$$

then the solution $(u(t,x), v(t,x))$ of (1.2) with $(u(0,x), v(0,x)) = (u_0, v_0)$ satisfies

$$\inf_{(\Phi_1,\Phi_2)\in\mathcal{G}(\alpha,\beta)} \|(u(t,\cdot), v(t,\cdot)) - (\Phi_1, \Phi_2)\|_X < \varepsilon \ \text{ for all } t \geq 0.$$

Proof. By contradiction, we assume that the assertion is false. Then there exist $\varepsilon > 0$ and two sequences $\{(u_n(0,x), v_n(0,x))\} \subset X$ and $\{t_n\} \subset \mathbb{R}$ such that

$$\inf_{(\Phi_1,\Phi_2)\in\mathcal{G}(\alpha,\beta)} \left\|\left(u_n(0,x), v_n(0,x)\right) - (\Phi_1, \Phi_2)\right\|_X < \frac{1}{n} \tag{2.30}$$

and

$$\inf_{(\Phi_1,\Phi_2)\in\mathcal{G}(\alpha,\beta)} \left\|\left(u_n(t_n,\cdot), v_n(t_n,\cdot)\right) - (\Phi_1, \Phi_2)\right\|_X \geq \varepsilon \text{ for any } n \in \mathbb{N}, \tag{2.31}$$

where $(u_n(t,x), v_n(t,x))$ is the solution of (1.2) with the initial data $(u_n(0,x), v_n(0,x))$. By (2.30) we have that $(u_n(0,x), v_n(0,x))$ converges to an element $(\Psi, \Theta) \in \mathcal{G}(\alpha,\beta)$ in X-norm. Since $Q(\Psi) = \alpha$, $Q(\Theta) = \beta$, and $E(\Psi,\Theta) = J(\alpha,\beta)$, by the conservation laws, we have

$$Q(u_n(t_n,\cdot)) = Q(u_n(0,x)) \to \alpha, \qquad Q(u_n(t_n,\cdot)) = Q(u_n(0,x)) \to \beta,$$

$$E\left(u_n(t_n,\cdot), v_n(t_n,\cdot)\right) = E(u_n(0,x), v_n(0,x)) \longrightarrow J(\alpha,\beta)$$

as $n \to \infty$. Let $\{a_n\}$ and $\{b_n\}$ be such that

$$Q(a_n u_n(0,x)) = a_n^2 Q(u_n(0,x)) = \alpha \text{ and } Q(b_n v_n(0,x)) = b_n^2 Q(v_n(0,x)) = \beta, \qquad n \in \mathbb{N}.$$

Set $\widetilde{u}_n(x) = a_n u_n(t_n, x)$ and $\widetilde{v}_n(x) = b_n v_n(t_n, x)$. It is clear that $Q(\widetilde{u}_n) = \alpha$ and $Q(\widetilde{v}_n) = \beta$, and since $a_n, b_n \to 1$, we have $\lim_{n\to\infty} E(\widetilde{u}_n, \widetilde{v}_n) = J(\alpha, \beta)$. Hence $\{(\widetilde{u}_n, \widetilde{v}_n)\}$ is a minimizing sequence for $J(\alpha,\beta)$. Thus, by Theorem 2.2(ii), for n large enough, there exists $(\Psi_n, \Theta_n) \in \mathcal{G}(\alpha,\beta)$ such that $\|(\widetilde{u}_n, \widetilde{v}_n) - (\Psi_n, \Theta_n)\|_X < \frac{\varepsilon}{2}$. Since

$$\begin{aligned} &\|(u_n(t_n,\cdot), v_n(t_n,\cdot)) - (\Psi_n, \Theta_n)\|_X \\ &\le \|(u_n(t_n,\cdot), v_n(t_n,\cdot)) - (\widetilde{u}_n, \widetilde{v}_n)\|_X + \|(\widetilde{u}_n, \widetilde{v}_n) - (\Psi_n, \Theta_n)\|_X , \end{aligned}$$

then, by (2.31), and $\|(u_n(t_n,\cdot), v_n(t_n,\cdot)) - (\widetilde{u}_n, \widetilde{v}_n)\|_X \underset{n\to\infty}{\to} 0$, we get
$\varepsilon \le \lim_{n\to\infty} \|(u_n(t_n,\cdot), v_n(t_n,\cdot)) - (\Psi_n, \Theta_n)\|_X \le \frac{\varepsilon}{2}$, which is a contradiction. □

Proof of Theorem 2.4. Firstly, by Proposition 2.11, for any fixed $\omega > \frac{\gamma^2}{N^2}$ and $\theta_1, \theta_2 \in \mathbb{R}$

$$\left(e^{i\theta_1} \sqrt{\frac{b-c}{b^2-ac}} \Phi_{\omega,\gamma}(\cdot), e^{i\theta_2} \sqrt{\frac{b-a}{b^2-ac}} \Phi_{\omega,\gamma}(\cdot) \right) \in \mathcal{G}(\alpha,\beta).$$

Using Proposition 2.11 again (see also (2.17)), for any $(\Phi_1, \Phi_2) \in \mathcal{G}(\alpha,\beta)$ we have

$$J(\alpha,\beta) = E(\Phi_1, \Phi_2) \ge E_1(\Phi_1) + E_2(\Phi_2) \ge J_1(\alpha) + J_2(\beta) = J(\alpha,\beta).$$

Therefore,

$$\frac{a}{2} \|\Phi_1\|_4^4 + \frac{c}{2} \|\Phi_2\|_4^4 + b \int_\Gamma |\Phi_1|^2 |\Phi_2|^2 \, dx = \frac{b^2-ac}{2(b-c)} \|\Phi_1\|_4^4 + \frac{b^2-ac}{2(b-a)} \|\Phi_2\|_4^4 ,$$

which implies that

$$\int_\Gamma \left(\sqrt{\frac{b-a}{b-c}} |\Phi_1|^2 - \sqrt{\frac{b-c}{b-a}} |\Phi_2|^2 \right)^2 dx = 0.$$

Thus,

$$\left(\frac{b-a}{b-c}\right)^{1/4} |\Phi_1(x)| = \left(\frac{b-c}{b-a}\right)^{1/4} |\Phi_2(x)| . \tag{2.32}$$

Secondly, notice that for $(\Phi_1, \Phi_2) \in \mathcal{G}(\alpha,\beta)$ there exist Lagrange multipliers $\omega_1, \omega_2 \in \mathbb{R}$ such that (Φ_1, Φ_2) is a solution of

$$\begin{cases} -\Delta_\gamma \Phi_1 + \omega_1 \Phi_1 = a |\Phi_1|^2 \Phi_1 + b |\Phi_2|^2 \Phi_1 \\ -\Delta_\gamma \Phi_2 + \omega_2 \Phi_2 = b |\Phi_1|^2 \Phi_2 + c |\Phi_2|^2 \Phi_2 \end{cases}$$

in a weak sense. Moreover, using (2.32), we simplify the system

$$\begin{cases} -\Delta_\gamma \Phi_1 + \omega_1 \Phi_1 = \frac{b^2-ac}{b-c} |\Phi_1|^2 \Phi_1 \\ -\Delta_\gamma \Phi_2 + \omega_2 \Phi_2 = \frac{b^2-ac}{b-a} |\Phi_2|^2 \Phi_2. \end{cases} \tag{2.33}$$

Following the arguments from the proofs of [17, Lemmas 25 and 26], we can show that the only pair of L^2-solution to (2.33) is given by

$$\Phi_1(x) = e^{i\theta_1} \sqrt{\frac{2\omega_1(b-c)}{b^2-ac}} \Phi_{\omega_1,\gamma}, \qquad \Phi_2(x) = e^{i\theta_2} \sqrt{\frac{2\omega_2(b-a)}{b^2-ac}} \Phi_{\omega_2,\gamma},$$

where $\theta_1, \theta_2 \in \mathbb{R}$. From (2.32) we get

$$|\Phi_{\omega_2,\gamma}(x)| = \left| \sqrt{\frac{\omega_1}{\omega_2}} \Phi_{\omega_1,\gamma}(x) \right| .$$

Finally, since

$$\|\Phi_1\|_2^2 = \alpha = 2\frac{b-c}{b^2-ac}(N\sqrt{\omega}-\gamma), \qquad \|\Phi_2\|_2^2 = \beta = 2\frac{b-a}{b^2-ac}(N\sqrt{\omega}-\gamma),$$

and using (2.32), it follows that $\omega_1 = \omega_2 = \omega$. This completes the proof. □

To end this section notice that Corollary 2.6 follows from Corollary 2.18 and Theorem 2.4.

3. Orbital stability of standing waves: Case of generalized nonlinearity

In this section we study orbital stability of standing waves of the general system

$$\begin{cases} i\partial_t u(t,x) + \Delta_\gamma u(t,x) + \left(a|u(t,x)|^{q-2} + b|v(t,x)|^p|u(t,x)|^{p-2}\right) u(t,x) = 0 \\ i\partial_t v(t,x) + \Delta_\gamma v(t,x) + \left(c|v(t,x)|^{r-2} + b|u(t,x)|^p|v(t,x)|^{p-2}\right) v(t,x) = 0 \\ (u(0,x), v(0,x)) = (u_0(x), v_0(x)). \end{cases} \tag{3.1}$$

We assume that $a, b, c \in \mathbb{R}$ and $2 < q, r, 2p$. The system is locally well posed (the proof follows analogously to [14, Theorem 4.10.1], see also [3, Section 2]). In particular, the conserved energy is given by

$$E(u,v) = \|u'\|_2^2 + \|v'\|_2^2 - \gamma\left(|u_1(0)|^2 + |v_1(0)|^2\right) - G(u,v), \tag{3.2}$$

where

$$G(u,v) = \frac{2a}{q}\|u\|_q^q + \frac{2c}{r}\|v\|_r^r + \frac{2b}{p}\|uv\|_p^p.$$

Remark 3.1. Notice that for $q, r, 2p < 6$ the global well-posedness holds. Due to the conservation of masses (2.1) and energy, we have

$$\begin{aligned} \|(u,v)\|_X^2 &= E(u,v) + \gamma\left(|u_1(0)|^2 + |v_1(0)|^2\right) + G(u,v) + \|u\|_2^2 + \|v\|_2^2 \\ &= \|(u_0,v_0)\|_X^2 - \gamma\left(|u_{01}(0)|^2 + |v_{01}(0)|^2\right) - G(u_0,v_0) + \gamma\left(|u_1(0)|^2 + |v_1(0)|^2\right) + G(u,v). \end{aligned} \tag{3.3}$$

Observe that

$$\begin{aligned} G(u,v) &\le C_1\|u'\|_2^{\frac{q-2}{2}}\|u\|_2^{\frac{q+2}{2}} + C_2\|v'\|_2^{\frac{r-2}{2}}\|v\|_2^{\frac{r+2}{2}} + \frac{2b}{p}\|u\|_{2p}^p\|v\|_{2p}^p \\ &\le C_1\|u'\|_2^{\frac{q-2}{2}}\|u\|_2^{\frac{q+2}{2}} + C_2\|v'\|_2^{\frac{r-2}{2}}\|v\|_2^{\frac{r+2}{2}} + \frac{b}{p}\left(\|u\|_{2p}^{2p} + \|v\|_{2p}^{2p}\right) \\ &\le \varepsilon\left(\|u'\|_2^2 + \|v'\|_2^2\right) + C_3(\varepsilon)\left(\|u\|_2^{\frac{2(q+2)}{6-q}} + \|v\|_2^{\frac{2(r+2)}{6-r}} + \|u\|_2^{\frac{2(2p+2)}{6-2p}} + \|v\|_2^{\frac{2(2p+2)}{6-2p}}\right). \end{aligned} \tag{3.4}$$

The above estimate is induced by Gagliardo–Nirenberg inequality (1.6) and the Young inequality $fg \le \varepsilon f^l + C_\varepsilon g^{l'}, \frac{1}{l} + \frac{1}{l'} = 1, l, l' > 1,\ f, g \ge 0$. Observe that the key point is that $l = \frac{4}{q-2} > 1$ for $2 < q < 6$ (analogously for r and $2p$). From (2.7), (3.3), and (3.4) we get

$$\|(u,v)\|_X^2 \le \frac{1}{1-\varepsilon}\left\{\|(u_0,v_0)\|_X^2 - G(u_0,v_0) + C(\|u_0\|_2^2, \|v_0\|_2^2)\right\},$$

i.e. the norm $\|(u,v)\|_X^2$ is controlled by the constant independent on time. Finally, repeating the proof of [14, Theorem 3.4.1] (starting from formula (3.4.2)), we obtain the global existence.

3.1. Stability of one component standing waves

We consider the standing waves of the simplest form $(e^{i\omega t}\Phi_1, 0)$ and $(0, e^{i\omega t}\Phi_2)$. Throughout this section, we assume additionally $p \geq 2$. Observe that $(\Phi_1, 0), (0, \Phi_2)$ are critical points of the functional

$$S_\omega(u,v) = \frac{1}{2}\left\{E(u,v) + \omega(\|u\|_2^2 + \|v\|_2^2)\right\},$$

that is, they satisfy Eq. (1.4). The description of the solutions to the first equation in (1.4) was given in [3, Theorem 4] (to obtain description of solutions to the second equation we need to replace q by r, and a by c):

Theorem 3.2. *Let $[s]$ denote the integer part of $s \in \mathbb{R}, \gamma \neq 0$, and $a > 0$. Then first equation in (1.4) has $\left[\frac{N-1}{2}\right] + 1$ (up to permutations of the edges of Γ and rotation) vector solutions $\Phi_k^\gamma = \left(\varphi_{k,j}^\gamma\right)_{j=1}^N, k = 0, \ldots, \left[\frac{N-1}{2}\right]$, which are given by*

$$\varphi_{k,j}^\gamma(x) = \begin{cases} \left[\frac{q\omega}{2a}\operatorname{sech}^2\left(\frac{(q-2)\sqrt{\omega}}{2}x - a_k\right)\right]^{\frac{1}{q-2}}, & j = 1, \ldots, k; \\ \left[\frac{q\omega}{2a}\operatorname{sech}^2\left(\frac{(q-2)\sqrt{\omega}}{2}x + a_k\right)\right]^{\frac{1}{q-2}}, & j = k+1, \ldots, N, \end{cases}$$

$$\textit{where } a_k = \tanh^{-1}\left(\frac{\gamma}{(N-2k)\sqrt{\omega}}\right), \textit{ and } \omega > \frac{\gamma^2}{(N-2k)^2}.$$

Below we deal with two types of instability: orbital and spectral. General definition of the orbital stability for a Hamiltonian system invariant under the action of some Lie group can be found in [23, Section 2]. The definition of the spectral instability involves a linearization of (3.1) around the profile of the standing wave. After making necessary technical steps we give precise Definition 3.3.

First, we linearize system (3.1) around $(\Phi_k^\gamma, 0)$. We observe that system (3.1) can be written in the form

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\mathcal{J}E'[u,v], \quad \mathcal{J} = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix},$$

and put

$$(u(t), v(t)) = e^{i\omega t}\left\{(\Phi_k^\gamma, 0) + (h_1(t), h_2(t))\right\}.$$

Since $S'_\omega(\Phi_k^\gamma, 0) = 0$, we get

$$\frac{d}{dt}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \mathcal{J}S''_\omega(\Phi_k^\gamma, 0)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + O\left(\|(h_1,h_2)\|_X^2\right),$$

where

$$S''_\omega(\Phi_k^\gamma, 0) = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \quad S_1 h_1 = -\hat{\Delta}_\gamma h_1 + \omega h_1 - a\left(\Phi_k^\gamma\right)^{q-2} h_1 - a(q-2)\left(\Phi_k^\gamma\right)^{q-2}\operatorname{Re}(h_1),$$

$$S_2 h_2 = \begin{cases} -\hat{\Delta}_\gamma h_2 + \omega h_2 - b\left(\Phi_k^\gamma\right)^2 h_2, \ p = 2 \\ -\hat{\Delta}_\gamma h_2 + \omega h_2, \ p > 2. \end{cases}$$

Here $-\hat{\Delta}_\gamma : H^1(\Gamma) \to \left(H^1(\Gamma)\right)^*$ is the unique bounded operator associated with the bounded on $H^1(\Gamma)$ bilinear form $t_\gamma(u_1, u_2) = (u_1', u_2')_2 - \gamma\operatorname{Re}\left(u_{11}(0)\overline{u_{21}(0)}\right)$. By the Representation Theorem [27, Chapter VI, Theorem 2.1], we can associate with the bilinear form

$$b_\gamma\big((h_1,h_2),(z_1,z_2)\big) = \langle S''_\omega(\Phi_k^\gamma, 0)(h_1,h_2),(z_1,z_2)\rangle_{X^*\times X}$$

self-adjoint in $L^2(\Gamma)\times L^2(\Gamma)$ (with the real scalar product) operator

$$L=\begin{pmatrix} L_1 & 0\\ 0 & L_2\end{pmatrix},\quad L_1h_1=-h_1''+\omega h_1-a\,(\Phi_k^\gamma)^{q-2}\,h_1-a(q-2)\,(\Phi_k^\gamma)^{q-2}\,\mathrm{Re}(h_1),$$

$$L_2h_2=\begin{cases}-h_2''+\omega h_2-b\,(\Phi_k^\gamma)^2\,h_2,\ p=2\\ -h_2''+\omega h_2,\ p>2.\end{cases},\quad \mathrm{dom}(L_1)=\mathrm{dom}(L_2)=\mathrm{dom}(\Delta_\gamma).$$

Putting $h_j=h_j^R+ih_j^I, j=1,2$, and identifying $L^2_{\mathbb{C}}(\Gamma)$ with $L^2_{\mathbb{R}}(\Gamma)\oplus L^2_{\mathbb{R}}(\Gamma)$, we conclude

$$\mathcal{J}L\quad\Longleftrightarrow\begin{pmatrix}0 & I_{L^2_{\mathbb{R}}} & 0 & 0\\ -I_{L^2_{\mathbb{R}}} & 0 & 0 & 0\\ 0 & 0 & 0 & I_{L^2_{\mathbb{R}}}\\ 0 & 0 & -I_{L^2_{\mathbb{R}}} & 0\end{pmatrix}\begin{pmatrix}L_1^R & 0 & 0 & 0\\ 0 & L_1^I & 0 & 0\\ 0 & 0 & L_2^R & 0\\ 0 & 0 & 0 & L_2^I\end{pmatrix}, \tag{3.5}$$

that is,

$$\mathcal{J}L\quad\Longleftrightarrow\begin{pmatrix}J & 0\\ 0 & J\end{pmatrix}\begin{pmatrix}L_1 & 0\\ 0 & L_2\end{pmatrix},$$

where $J=\begin{pmatrix}0 & I_{L^2_{\mathbb{R}}}\\ -I_{L^2_{\mathbb{R}}} & 0\end{pmatrix}$ and

$$L_1^R=-\frac{d^2}{dx^2}+\omega-a(q-1)(\Phi_k^\gamma)^{q-2},\quad L_1^I=-\frac{d^2}{dx^2}+\omega-a(\Phi_k^\gamma)^{q-2},$$
$$L_2^R=L_2^I=L_2,\quad \mathrm{dom}(L_j^R)=\mathrm{dom}(L_j^I)=\mathrm{dom}(\Delta_\gamma),\ j=1,2.$$

Definition 3.3. The standing wave $e^{i\omega t}(\Phi_k^\gamma,0)$ is said to be spectrally unstable if there exist λ with $\mathrm{Re}\,\lambda>0$ and $\vec{h}=(h_1,h_2)\in\mathrm{dom}\,(\Delta_\gamma)\times\mathrm{dom}(\Delta_\gamma)$ such that

$$\mathcal{J}L\vec{h}=\lambda\vec{h}.$$

The notion of spectral instability is particularly important since frequently its presence leads to nonlinear instability.

Remark 3.4. (i) In a view of (3.5) the above definition can be rewritten as

$$\begin{pmatrix}J & 0\\ 0 & J\end{pmatrix}\begin{pmatrix}L_1 & 0\\ 0 & L_2\end{pmatrix}\begin{pmatrix}h_1\\ h_2\end{pmatrix}=\lambda\begin{pmatrix}h_1\\ h_2\end{pmatrix}.$$

(ii) Notice that spectral instability implies that $(0,0)$ is unstable solution to the linearized equation

$$\frac{d}{dt}\begin{pmatrix}u(t)\\ v(t)\end{pmatrix}=\mathcal{J}S''_\omega\,(\Phi_k^\gamma,0)\begin{pmatrix}u(t)\\ v(t)\end{pmatrix}$$

in the sense of Lyapunov.

Below we list spectral properties of the operators L_1^R, L_1^I proved in [3, Proposition 6.1] and [26, Theorem 3.1]: let $k=0,\dots,\left[\frac{N-1}{2}\right]$, then

$a)$ $\ker(L_1^I)=\mathrm{span}\{\Phi_k^\gamma\}$ and $\ker(L_1^I)=\{0\}$;

b) $L_1^I \geq 0$ and $n\left(L_1^R\right) = \begin{cases} k+1 & \text{if} \quad \gamma > 0, \\ N-k & \text{if} \quad \gamma < 0 \end{cases}$;

c) $\sigma_{\text{ess}}(L_1^R) = \sigma_{\text{ess}}(L_1^I) = [\omega, \infty)$.

We use these properties to show the principal result of this subsection.

Theorem 3.5. *Let $a > 0$, $q > 2$, and Φ_k^γ be defined in Theorem 3.2.*

(i) Assume $p \geq 2$. If either $\gamma > 0, k \geq 1$ or $\gamma < 0, k \geq 0$, then $e^{i\omega t}(\Phi_k^\gamma, 0)$ is spectrally unstable. If additionally $p, q > 3$, then we have orbital instability.

(ii) Assume $p > 2$. If $\gamma > 0, k = 0$, then $e^{i\omega t}(\Phi_0^\gamma, 0)$ is orbitally stable for $2 < q \leq 6$. Moreover, for $q > 6$ there exists $\omega_1 > \frac{\gamma^2}{N^2}$ such that $e^{i\omega t}(\Phi_0^\gamma, 0)$ is orbitally stable as $\omega \in (\frac{\gamma^2}{N^2}, \omega_1)$, while it is orbitally unstable as $\omega > \omega_1$.

Proof. (i) From [21, Theorem 1.2] one concludes that the operator JL_1 has a positive eigenvalue for $\gamma > 0, k \geq 1$, and $\gamma < 0, k \geq 0$. Indeed, $JL_1\vec{w} = \lambda\vec{w}, \vec{w} = (w_1, w_2)$, is equivalent to

$$\begin{pmatrix} 0 & L_1^I \\ -L_1^R & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \lambda \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{3.6}$$

or $\begin{cases} L_1^I w_2 = \lambda w_1 \\ L_1^R w_1 = -\lambda w_2 \end{cases}$. Since $\text{ran}(L_1^I) \perp \ker(L_1^I)$, we get $w_1 \in \ker(L_1^I)^\perp$. Hence

$$w_2 = \lambda (L_1^I)^{-1} w_1, \quad L_1^R w_1 + \lambda^2 (L_1^I)^{-1} w_1 = 0.$$

Consider the problem

$$PL_1^R w_1 + \lambda^2 P (L_1^I)^{-1} w_1 = 0,$$

where P is the orthogonal projection onto

$$\ker(L_1^I)^\perp = \left\{(v_1, v_2) \in L^2(\Gamma) \times L^2(\Gamma) : \left((v_1, v_2), (\Phi_k^\gamma, 0)\right)_2 = 0\right\}$$

(here we assume that $L^2(\Gamma)$ is endowed with the usual complex inner product). The projection P serves to fit the problem into the Hilbert space $\ker(L_1^I)^\perp$. [21, Theorem 1.2] states that the number $I(L_1^R, L_1^I)$ of positive λ satisfying (3.6) is estimated by

$$n(PL_1^R) - n(P(L_1^I)^{-1}) \leq I(L_1^R, L_1^I). \tag{3.7}$$

Using spectral properties $a), b)$ mentioned before the theorem, formula (3.7) yields

$$I(L_1^R, L_1^I) \geq \begin{cases} k, & \gamma > 0, \\ N-k-1, & \gamma < 0. \end{cases}$$

By (3.5), there exists $\lambda > 0$ satisfying Definition 3.3 under the assumptions in (i), that is, $(\Phi_0^\gamma, 0)$ is spectrally unstable. Using the fact that the nonlinearity

$$F_{p,q,r}(u, v) = (a|u|^{q-2}u + b|v|^p|u|^{p-2}u, c|v|^{r-2}v + b|u|^p|v|^{p-2}v)$$

is of C^2 class for $p, q > 3$, we conclude that mapping data-solution associated to (3.1) is of class C^2 (for instance, see [19, Step 4 of proof of Proposition 2.1]). Finally, to imply the orbital instability from the spectral one, the approach by [24] (see Theorem 2 and [19, Theorem 5.8, Corollary 5.9]) can be used.

(*ii*) By $L_2 \geq \omega$, we deduce $n(L) = n(L_1) = 1$. Observe that $S''_\omega(\Phi_0^\gamma, 0)$ satisfies [45, Condition (G)]. Indeed, for $\vec{v} = (v_1, v_2) \in X$

$$\begin{aligned}\langle S''_\omega(\Phi_0^\gamma, 0)\vec{v}, \vec{v}\rangle_{X^* \times X} &= \int_\Gamma \Big\{|v_1'|^2 + |v_2'|^2 + \omega\Big(|v_1|^2 + |v_2|^2\Big)\Big\}dx \\ &- \int_\Gamma \Big\{a(\Phi_0^\gamma)^{q-2}|v_1|^2 + a(q-2)(\Phi_0^\gamma)^{q-2}(\mathrm{Re}\, v_1)^2\Big\}dx - \gamma\Big(|v_{11}(0)|^2 + |v_{21}(0)|^2\Big) \\ &\geq \min\{\frac{1}{2}, \omega\}\|\vec{v}\|_X^2 - \Big((q-1)aM^{q-2} + \frac{2\gamma^2}{N^2}\Big)\|\vec{v}\|_2^2,\end{aligned}$$

where $M = \|\Phi_0^\gamma\|_\infty$ and we have used

$$-\gamma\Big(|v_{11}(0)|^2 + |v_{21}(0)|^2\Big) + \frac{1}{2}\int_\Gamma \Big\{|v_1'|^2 + |v_2'|^2\Big\}dx \geq -\frac{2\gamma^2}{N^2}\|\vec{v}\|_2^2.$$

Let $\mathcal{R} : X \to X^*$ be the Riesz isomorphism. Thus, from [45, Lemma 5.4] we conclude $\sigma_{\mathrm{ess}}(\mathcal{R}^{-1}S''_\omega(\Phi_0^\gamma, 0)) = \sigma_{\mathrm{ess}}(L) = [\omega, \infty)$, $\ker(\mathcal{R}^{-1}S''_\omega(\Phi_0^\gamma, 0)) = \ker(L) = \mathrm{span}\{i(\Phi_0^\gamma, 0)\}$, and $n(\mathcal{R}^{-1}S''_\omega(\Phi_0^\gamma, 0)) = n(L) = 1$.

Denote $d''(\omega) = \partial_\omega^2(S_\omega(\Phi_0^\gamma, 0))$. By [6, Proposition 3.19-(*i*)], we get $d''(\omega) > 0$ for $2 < q \leq 6$, and for $q > 6$ there exists ω_1 such that $d''(\omega) > 0$ as $\omega \in (\frac{\gamma^2}{N^2}, \omega_1)$, and $d''(\omega) < 0$ as $\omega > \omega_1$. Finally, the result follows applying [22, Theorem 3]. □

Remark 3.6. (*i*) Observe that for $p = 2, q = 4, \gamma > 0, k = 0$, and $b < a$ we have $L_2 > L_1^I$, then $\sigma(L_2) = [\omega, \infty)$. Therefore, $n(\mathcal{R}^{-1}S''_\omega(\Phi_0^\gamma, 0)) = 1$. As in the proof of item (*ii*), we have $d''(\omega) > 0$, then the orbital stability of $e^{i\omega t}(\Phi_0^\gamma, 0)$ holds.

(*ii*) All the above results on the stability/instability of $(\Phi_k^\gamma, 0)$ can be repeated for the profile $(0, \Phi_2) = (0, \Phi_k^\gamma)$ with q replaced by r and a replaced by c.

3.2. Stability of two component standing waves

In this subsection we study stability properties of the standing waves of system (3.1) of the form $(e^{i\omega t}\Phi(x), e^{i\omega t}\Phi(x))$ in the case $q = r = 2p-2, a = c = 1$:

$$\begin{cases} i\partial_t u(t,x) + \Delta_\gamma u(t,x) + \Big(|u(t,x)|^{2p-2} + b|v(t,x)|^p|u(t,x)|^{p-2}\Big)u(t,x) = 0 \\ i\partial_t v(t,x) + \Delta_\gamma v(t,x) + \Big(|v(t,x)|^{2p-2} + b|u(t,x)|^p|v(t,x)|^{p-2}\Big)v(t,x) = 0. \end{cases} \tag{3.8}$$

It is easily seen that $\Phi(x)$ solves

$$-\Delta_\gamma \Phi + \omega\Phi - (b+1)\Phi^{2p-1} = 0. \tag{3.9}$$

The description of solutions to (3.9) is given by Theorem 3.2 (one just needs to substitute q by $2p$ and a by $b+1$). Namely, for $b > -1$ we have $\left[\frac{N-1}{2}\right] + 1$ solutions $\Phi_k^\gamma = \left(\varphi_{k,j}^\gamma\right)_{j=1}^N$, $k = 0, \ldots, \left[\frac{N-1}{2}\right]$, of the form

$$\varphi_{k,j}^\gamma(x) = \begin{cases} \left[\frac{p\omega}{(b+1)}\,\mathrm{sech}^2\Big((p-1)\sqrt{\omega}x - a_k\Big)\right]^{\frac{1}{2p-2}}, \quad j = 1, \ldots, k; \\ \left[\frac{p\omega}{(b+1)}\,\mathrm{sech}^2\Big((p-1)\sqrt{\omega}x + a_k\Big)\right]^{\frac{1}{2p-2}}, \quad j = k+1, \ldots, N, \end{cases} \tag{3.10}$$
$$\text{where } a_k = \tanh^{-1}\left(\frac{\gamma}{(N-2k)\sqrt{\omega}}\right), \text{ and } \omega > \frac{\gamma^2}{(N-2k)^2}.$$

The main results of this subsection are the following two theorems. The first one deals with arbitrary k and the second one contains stability results for the particular case of symmetric tail-profile $(\Phi_0^\gamma, \Phi_0^\gamma)$.

Theorem 3.7. *Let $p \geq 2, b > -1$, and $(\Phi_k^\gamma, \Phi_k^\gamma)$ be defined by (3.10). If $\gamma > 0, k \geq 2$ or $\gamma < 0, N-k \geq 3$, then the standing wave $e^{i\omega t}(\Phi_k^\gamma, \Phi_k^\gamma)$ is spectrally unstable. Moreover, for $p > 3$ the orbital instability holds.*

If additionally $0 < b < p-1$, then the above assertions hold for $\gamma < 0, k \geq 0$ and $\gamma > 0, k = 1$.

Theorem 3.8. *Let $\gamma > 0, k = 0$, then for $2 \leq p < 3$ and $0 < b \neq p-1$ the standing wave $e^{i\omega t}(\Phi_0^\gamma, \Phi_0^\gamma)$ is orbitally stable, while for $p > 3$ and $b > p-1$ the standing wave $e^{i\omega t}(\Phi_0^\gamma, \Phi_0^\gamma)$ is orbitally unstable.*

As in the previous case, we linearize (3.8) at $(\Phi_k^\gamma, \Phi_k^\gamma)$. Notice that the profile $(\Phi_1(x), \Phi_2(x))$ of the standing wave $(e^{i\omega_1 t}\Phi_1(x), e^{i\omega_2 t}\Phi_2(x))$ is the critical point of the action functional given by

$$S_{\omega_1,\omega_2}(u,v) = \frac{1}{2}\left\{E(u,v) + \omega_1\|u\|_2^2 + \omega_2\|v\|_2^2\right\},$$

where E is defined by (3.2). We obtain for $\vec{h} = (h_1, h_2) \in X$

$$S''_{\omega,\omega}(\Phi_k^\gamma, \Phi_k^\gamma)\vec{h} = \begin{pmatrix} \tilde{S}_1\vec{h} \\ \tilde{S}_2\vec{h} \end{pmatrix}, \ \tilde{S}_1\vec{h} = \tilde{l_\gamma}h_1 - bp(\Phi_k^\gamma)^{2p-2}\operatorname{Re}(h_2), \ \tilde{S}_2\vec{h} = \tilde{l_\gamma}h_2 - bp(\Phi_k^\gamma)^{2p-2}\operatorname{Re}(h_1),$$
$$\tilde{l_\gamma} = -\hat{\Delta}_\gamma + \left(\omega - (1+b)(\Phi_k^\gamma)^{2p-2}\right) - \left(\big(2p-2+b(p-2)\big)(\Phi_k^\gamma)^{2p-2}\right)\operatorname{Re}(\cdot).$$

Analogously to the previous case, we associate with the bilinear form $\left\langle S''_{\omega,\omega}(\Phi_k^\gamma, \Phi_k^\gamma)(h_1,h_2),(z_1,z_2)\right\rangle_{X^*\times X}$ the self-adjoint in $L^2(\Gamma)\times L^2(\Gamma)$ operator $\tilde{L}$. Putting $h_j = h_j^R + ih_j^I, j = 1,2$, we obtain

$$\tilde{L} = \tilde{L}^R\vec{h}^R + i\tilde{L}^I\vec{h}^I, \quad \tilde{L}^I = \begin{pmatrix} \tilde{L}_1^I & 0 \\ 0 & \tilde{L}_2^I \end{pmatrix}, \ \tilde{L}_j^I = -\frac{d^2}{dx^2} + \omega - (1+b)(\Phi_k^\gamma)^{2p-2},$$
$$\tilde{L}^R = \begin{pmatrix} -\frac{d^2}{dx^2} + \omega - \big(2p-1+b(p-1)\big)(\Phi_k^\gamma)^{2p-2} & -bp(\Phi_k^\gamma)^{2p-2} \\ -bp(\Phi_k^\gamma)^{2p-2} & -\frac{d^2}{dx^2} + \omega - \big(2p-1+b(p-1)\big)(\Phi_k^\gamma)^{2p-2} \end{pmatrix}, \tag{3.11}$$

where $\operatorname{dom}(\tilde{L}^I) = \operatorname{dom}(\tilde{L}^R) = \operatorname{dom}(\Delta_\gamma)\times\operatorname{dom}(\Delta_\gamma)$. It is easily seen that

$$\mathcal{J}\tilde{L} \iff \begin{pmatrix} 0 & I_{L^2_{\mathbb{R}}\times L^2_{\mathbb{R}}} \\ -I_{L^2_{\mathbb{R}}\times L^2_{\mathbb{R}}} & 0 \end{pmatrix}\begin{pmatrix} \tilde{L}^R & 0 \\ 0 & \tilde{L}^I \end{pmatrix} = \begin{pmatrix} 0 & \tilde{L}^I \\ -\tilde{L}^R & 0 \end{pmatrix}. \tag{3.12}$$

Remark 3.9. The definition of the spectral instability is analogous to Definition 3.3. One just needs to substitute L by $\tilde{L}$. By (3.12), it means the existence of λ with $\operatorname{Re}\lambda > 0$ and $(h_1,h_2) \in \operatorname{dom}(\Delta_\gamma)\times\operatorname{dom}(\Delta_\gamma)$ such that

$$\begin{pmatrix} 0 & \tilde{L}^I \\ -\tilde{L}^R & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \lambda\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

Below we list some properties of the operators $\tilde{L}^I$ and $\tilde{L}^R$ (see [3, Proposition 6.1]):

a) $\ker(\tilde{L}^I) = \operatorname{span}\{(\Phi_k^\gamma, 0), (0, \Phi_k^\gamma)\}$;
b) $\tilde{L}^I \geq 0$;
c) $\sigma_{\mathrm{ess}}(\tilde{L}^I) = \sigma_{\mathrm{ess}}(\tilde{L}^R) = [\omega, \infty)$.

Remark 3.10. The property $\sigma_{\mathrm{ess}}(\tilde{L}^R)=[\omega,\infty)$ follows from the representation

$$\tilde{L}^R=\begin{pmatrix} -\frac{d^2}{dx^2}+\omega-\big(2p-1+b(p-1)\big)(\Phi_k^\gamma)^{2p-2} & 0\\ 0 & -\frac{d^2}{dx^2}+\omega-\big(2p-1+b(p-1)\big)(\Phi_k^\gamma)^{2p-2}\end{pmatrix} + \begin{pmatrix} 0 & -bp(\Phi_k^\gamma)^{2p-2}\\ -bp(\Phi_k^\gamma)^{2p-2} & 0\end{pmatrix}$$

and the fact that $-bp(\Phi_k^\gamma)^{2p-2}$ is relatively $-\Delta_\gamma$-compact.

Below we count $n(\tilde{L}^R)$.

Proposition 3.11. *Let $p>-1$, then $n(\tilde{L}^R)\geq\begin{cases} k+1, & \gamma>0\\ N-k, & \gamma<0\end{cases}$ with $k=0,\dots,[\frac{N-1}{2}]$. Moreover, if additionally*

(a) $0<b<p-1$, then $n(\tilde{L}^R)\geq\begin{cases} k+2, & \gamma>0\\ N-k+1, & \gamma<0\end{cases}$;

(b) $b>p-1$, then $n(\tilde{L}^R)=\begin{cases} k+1, & \gamma>0\\ N-k, & \gamma<0.\end{cases}$

To prove the above proposition, we suppose that $\lambda\in\sigma_p(\tilde{L}^R)$ and (h_1,h_2) is the corresponding eigenvector. Then denoting $h_+=h_1+h_2$ and $h_-=h_1-h_2$, we get

$$\begin{cases} \tilde{L}_+^R h_+=\lambda h_+\\ \tilde{L}_-^R h_-=\lambda h_-\end{cases},\quad \tilde{L}_+^R=-\frac{d^2}{dx^2}+\omega-(2p-1)(b+1)(\Phi_k^\gamma)^{2p-2},$$
$$\tilde{L}_-^R=-\frac{d^2}{dx^2}+\omega-(2p-b-1)(\Phi_k^\gamma)^{2p-2},\quad \mathrm{dom}(\tilde{L}_+^R)=\mathrm{dom}(\tilde{L}_-^R)=\mathrm{dom}(\Delta_\gamma). \tag{3.13}$$

From (3.13) we conclude

$$(h_1,h_2)\in\ker(\tilde{L}^R)\quad\Longleftrightarrow\quad (h_+,h_-)\in\ker\begin{pmatrix}\tilde{L}_+^R & 0\\ 0 & \tilde{L}_-^R\end{pmatrix} \tag{3.14}$$

and

$$n(\tilde{L}^R)=n(\tilde{L}_+^R)+n(\tilde{L}_-^R). \tag{3.15}$$

Thus, it is sufficient to study the kernel and the Morse index of the operators $\tilde{L}_+^R$ and $\tilde{L}_-^R$.

Lemma 3.12. *Let $p>-1$ and the operator $\tilde{L}_+^R$ be defined by (3.13), then $\ker(\tilde{L}_+^R)=\{0\}$ and*

$$n(\tilde{L}_+^R)=\begin{cases} k+1, & \gamma>0\\ N-k, & \gamma<0\end{cases},\quad k=0,\dots,\left[\tfrac{N-1}{2}\right].$$

The proof follows from [3, Proposition 6.1] and [26, Theorem 3.1]. Next we evaluate $n(\tilde{L}_-^R)$. To do that we will use the ideas from [34]. Denote $\Psi_k^\gamma=\left(\psi_{k,j}^\gamma\right)_{j=1}^N$, $k=0,\dots,[\frac{N-1}{2}]$, where

$$\psi_{k,j}^\gamma(x)=\begin{cases}\left[p\,\mathrm{sech}^2\Big((p-1)x-a_k\Big)\right]^{\frac{1}{2p-2}}, & j=1,\dots,k;\\ \left[p\,\mathrm{sech}^2\Big((p-1)x+a_k\Big)\right]^{\frac{1}{2p-2}}, & j=k+1,\dots,N,\end{cases}$$
$$\text{with } a_k=\tanh^{-1}\left(\frac{\gamma}{(N-2k)\sqrt{\omega}}\right),\ \text{and } \omega>\frac{\gamma^2}{(N-2k)^2}. \tag{3.16}$$

We introduce the self-adjoint operator

$$\tilde{L}_\varepsilon = -\frac{d^2}{dx^2} + 1 - \varepsilon(\Psi_k^\gamma)^{2p-2}, \quad \mathrm{dom}(\tilde{L}_\varepsilon) = \mathrm{dom}(\Delta_{\frac{\gamma}{\sqrt{\omega}}}), \tag{3.17}$$

and prove the following technical lemma.

Lemma 3.13. *Let the operator $\tilde{L}_\varepsilon$ be defined by (3.17). Set*

$$L_k^2(\Gamma) = \left\{(v_e)_{e=1}^N \in L^2(\Gamma) : v_1(x) = \cdots = v_k(x), \quad v_{k+1}(x) = \cdots = v_N(x)\right\}.$$

Then the following assertions hold.

- (i) *For $\varepsilon < 1$ the operator $\tilde{L}_\varepsilon$ is positive definite.*
- (ii) *Let $1 < \varepsilon < 2p-1$, then the operator $\tilde{L}_\varepsilon$ is invertible and*
  - (a) *for $k = 0$ in $L^2(\Gamma)$ we get $n(\tilde{L}_\varepsilon) = 1$ as $\gamma > 0$, and $n(\tilde{L}_\varepsilon) \geq 1$ as $\gamma < 0$;*
  - (b) *for $k \geq 1$ in $L_k^2(\Gamma)$ we get $n(\tilde{L}_\varepsilon) = 1$ as $\gamma < 0$, and $n(\tilde{L}_\varepsilon) \geq 1$ as $\gamma > 0$.*

Proof. Let $\varepsilon = 1$, then it is easily seen that $\ker(\tilde{L}_1) = \mathrm{span}\{\Psi_k^\gamma\}$ and $\tilde{L}_1 \geq 0$. Then assertion (i) is trivial observing that for $\varepsilon < 1$ one gets $\tilde{L}_\varepsilon > \tilde{L}_1$.

Notice that $\sigma_{\mathrm{ess}}(\tilde{L}_\varepsilon) = [1, \infty)$, and for $\varepsilon > 1$ we get

$$(\tilde{L}_\varepsilon \Psi_k^\gamma, \Psi_k^\gamma)_2 = -(\varepsilon - 1)\|\Psi_k^\gamma\|_{2p}^{2p} < 0,$$

then the first eigenvalue of $\tilde{L}_\varepsilon$ is negative. In particular, by Min–Max theorem, the discrete spectrum of $\tilde{L}_\varepsilon$ moves to the left when ε increases. Let $\varepsilon = 2p-1$, then by [7, Proposition 4] and [6, Proposition 3.17], we get:

- $n(\tilde{L}_{2p-1}) = 1$ for $k = 0, \gamma > 0$ in $L^2(\Gamma)$ and for $k \geq 1, \gamma < 0$ in $L_k^2(\Gamma)$.
- $n(\tilde{L}_{2p-1}) = 2$ for $k = 0, \gamma < 0$ and for $k \geq 1, \gamma > 0$ in $L_k^2(\Gamma)$.

Using analyticity of the family $\tilde{L}_\varepsilon$, we conclude (a) and (b) for $1 < \varepsilon < 2p-1$. □

Lemma 3.14. *Let the operator $\tilde{L}_-^R$ be defined by (3.13). Then the following assertions hold.*

- (i) *For $b > p-1$ the operator $\tilde{L}_-^R$ is positive definite.*
- (ii) *Let $0 < b < p-1$, then the operator $\tilde{L}_-^R$ is invertible and*
  - (a) *for $k = 0$ in $L^2(\Gamma)$ we get $n(\tilde{L}_-^R) = 1$ as $\gamma > 0$, and $n(\tilde{L}_-^R) \geq 1$ as $\gamma < 0$;*
  - (b) *for $k \geq 1$ in $L_k^2(\Gamma)$ we get $n(\tilde{L}_-^R) = 1$ as $\gamma < 0$, and $n(\tilde{L}_-^R) \geq 1$ as $\gamma > 0$.*

Proof. Put $\varepsilon = \frac{2p-1-b}{b+1}$, then $1 < \varepsilon < 2p-1$ for $0 < b < p-1$ and $\varepsilon < 1$ for $b > p-1$. Let $\lambda \in \sigma_p(\tilde{L}_-^R)$ and $f(x) \in \mathrm{dom}(\Delta_\gamma)$ be the corresponding eigenvector: $\tilde{L}_-^R f = \lambda f$. Define $h(x) = f(\frac{x}{\sqrt{\omega}})$, then $h(x) \in \mathrm{dom}(\Delta_{\frac{\gamma}{\sqrt{\omega}}})$ and $\tilde{L}_\varepsilon h = \frac{\lambda}{\omega} h$. □

Proof of Proposition 3.11. The first assertion follows from formula (3.15) and Lemma 3.12. Assertions $(a), (b)$ follow from Lemma 3.14. □

Proof of Theorem 3.7. As in the proof of Theorem 3.5, we can show that there exist $\lambda > 0$ and $(w_1, w_2) \in \mathrm{dom}(\Delta_\gamma) \times \mathrm{dom}(\Delta_\gamma)$ such that

$$\begin{pmatrix} 0 & \tilde{L}^I \\ -\tilde{L}^R & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \lambda \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{3.18}$$

As before, we rely on [21, Theorem 1.2] which states that the number $I(\tilde{L}^R,\tilde{L}^I)$ of positive λ satisfying (3.18) is estimated by

$$n(P\tilde{L}^R)-n(P(\tilde{L}^I)^{-1})\leq I(\tilde{L}^R,\tilde{L}^I), \tag{3.19}$$

where P is the orthogonal projection onto

$$\ker(\tilde{L}^I)^\perp=\left\{(v_1,v_2)\in L^2(\Gamma)\times L^2(\Gamma):\big((v_1,v_2),(\Phi_k^\gamma,0)\big)_2=\big((v_1,v_2),(0,\Phi_k^\gamma)\big)_2=0\right\}$$

(here we assume that $L^2(\Gamma)$ is endowed with the usual complex inner product). Using positivity of $\tilde{L}^I$ and the estimates of $n(\tilde{L}^R)$ from Proposition 3.11, formula (3.19) yields:

- for $b>-1$: $I(\tilde{L}^R,\tilde{L}^I)\geq\begin{cases}k-1, & \gamma>0\\ N-k-2, & \gamma<0\end{cases}$
- for $0<b<p-1$: $I(\tilde{L}^R,\tilde{L}^I)\geq\begin{cases}k, & \gamma>0\\ N-k-1, & \gamma<0.\end{cases}$

As in the previous subsection, the orbital instability part follows from the fact that the nonlinearity $F_p(u,v)=\Big(|u|^{2p-2}u+b|v|^p|u|^{p-2}u,|v|^{2p-2}v+b|u|^p|v|^{p-2}v\Big)$ is of C^2 class for $p>3$. $\square$

Proof of Theorem 3.8. Our aim is to use [23, Stability Theorem and Instability Theorem]. Observe that [23, *Assumption 2*] follows by the Implicit Function Theorem. Indeed, consider the mapping $F((\omega_1,\omega_2),(u,v)):\mathbb{R}_+^2\times X_\mathbb{R}\to X_\mathbb{R}^*$, where $F((\omega_1,\omega_2),(u,v))=S'_{\omega_1,\omega_2}(u,v)$, and $X_\mathbb{R}$ is the Banach space consisting of real-valued functions from X. Observe that $\ker(S''_{\omega,\omega}(\Phi_0^\gamma,\Phi_0^\gamma))=\operatorname{span}\{(0,0)\}$ in $X_\mathbb{R}$ since in X we have $\ker(S''_{\omega,\omega}(\Phi_0^\gamma,\Phi_0^\gamma))=\operatorname{span}\{(i\Phi_0^\gamma,i\Phi_0^\gamma)\}$. Notice also that $S''_{\omega,\omega}(\Phi_0^\gamma,\Phi_0^\gamma)$ satisfies [45, Condition (G)]. Indeed, for $\vec{v}=(v_1,v_2)\in X$

$$\begin{aligned}&\langle S''_{\omega,\omega}(\Phi_0^\gamma,\Phi_0^\gamma)\vec{v},\vec{v}\rangle_{X^*\times X}=\int_\Gamma\Big\{|v_1'|^2+|v_2'|^2+\omega\Big(|v_1|^2+|v_2|^2\Big)\Big\}dx\\ &-\int_\Gamma\Big\{(1+b)(\Phi_0^\gamma)^{2p-2}\Big(|v_1|^2+|v_2|^2\Big)+\big(2p-2+b(p-2)\big)(\Phi_0^\gamma)^{2p-2}\Big((\operatorname{Re}v_1)^2+(\operatorname{Re}v_2)^2\Big)\Big\}dx\\ &-2bp\int_\Gamma(\Phi_0^\gamma)^{2p-2}\operatorname{Re}v_1\operatorname{Re}v_2dx-\gamma\Big(|v_{11}(0)|^2+|v_{21}(0)|^2\Big)\\ &\geq\min\{\frac{1}{2},\omega\}\|\vec{v}\|_X^2-\Big((1+b)(2p-1)M^{2p-2}+\frac{2\gamma^2}{N^2}\Big)\|\vec{v}\|_2^2,\quad M=\|\Phi_0^\gamma\|_\infty.\end{aligned}$$

Thus, by [45, Lemma 5.4], we conclude $\sigma_{\text{ess}}(\mathcal{R}^{-1}S''_{\omega,\omega}(\Phi_0^\gamma,\Phi_0^\gamma))=\sigma_{\text{ess}}(\tilde{L})=[\omega,\infty)$ since $\tilde{L}=\tilde{L}^R+i\tilde{L}^I$ and $\sigma_{\text{ess}}(\tilde{L}^R)=\sigma_{\text{ess}}(\tilde{L}^I)=[\omega,\infty)$, where $\mathcal{R}:X\to X^*$ is the Riesz isomorphism. Hence $S''_{\omega,\omega}(\Phi_0^\gamma,\Phi_0^\gamma):X_\mathbb{R}\to X_\mathbb{R}^*$ is invertible and bounded. Then, by the Implicit Function Theorem, there is an open neighborhood Ω of (ω,ω) and a unique function $(\Phi_1(\omega_1,\omega_2),\Phi_2(\omega_1,\omega_2)):\Omega\to X_\mathbb{R}$ such that $S'_{\omega_1,\omega_2}(\Phi_1(\omega_1,\omega_2),\Phi_2(\omega_1,\omega_2))=S'_{\omega,\omega}(\Phi_0^\gamma,\Phi_0^\gamma)=0$.

Next, let us check that [23, *Assumption* 3] holds. By (3.14), (3.15), Lemmas 3.12 and 3.14, we conclude that $\ker(\tilde{L}^R)=\{0\}$ for $b\neq p-1$, and $n(\tilde{L}^R)<\infty$, then, by (3.11) and [45, Lemma 5.4], we obtain $\ker(\mathcal{R}^{-1}S''_{\omega,\omega}(\Phi_0^\gamma,\Phi_0^\gamma))=\ker(\tilde{L})=\operatorname{span}\{i(\Phi_0^\gamma,0),i(0,\Phi_0^\gamma)\}$ and $n(\mathcal{R}^{-1}S''_{\omega,\omega}(\Phi_0^\gamma,\Phi_0^\gamma))=n(\tilde{L}^R)<\infty$.

Let $d''(\omega)$ be the Hessian of $S_{\omega_1,\omega_2}(\Phi_1(\omega_1,\omega_2),\Phi_2(\omega_1,\omega_2))$ at (ω,ω). It is easily seen that

$$d''(\omega)=\frac{1}{2}\begin{pmatrix}\partial_{\omega_1}\|\Phi_1(\omega_1,\omega_2)\|_2^2 & \partial_{\omega_2}\|\Phi_1(\omega_1,\omega_2)\|_2^2\\ \partial_{\omega_1}\|\Phi_2(\omega_1,\omega_2)\|_2^2 & \partial_{\omega_2}\|\Phi_2(\omega_1,\omega_2)\|_2^2\end{pmatrix}\Big|_{(\omega_1,\omega_2)=(\omega,\omega)}.$$

Denote by $p(d''(\omega))$ the number of positive eigenvalues of $d''(\omega)$. By [23, Stability Theorem and Instability Theorem], we conclude:

- if $n(\tilde{L}) = p(d''(\omega))$, then $e^{i\omega t}(\Phi_0^\gamma, \Phi_0^\gamma)$ is orbitally stable;
- if $n(\tilde{L}) - p(d''(\omega))$ is odd, then $e^{i\omega t}(\Phi_0^\gamma, \Phi_0^\gamma)$ is spectrally unstable.

Step 1. Note that $\partial_{\omega_j}\|\Phi_i(\omega_1,\omega_2)\|_2^2 = \int_\Gamma \Phi_i(\omega_1,\omega_2)\partial_{\omega_j}\Phi_i(\omega_1,\omega_2)dx$, $i,j = 1,2$. Differentiating $S'_{\omega,\omega}(\Phi_1(\omega_1,\omega_2), \Phi_2(\omega_1,\omega_2))$ with respect to ω_1, using the chain rule, and denoting $(h_1,h_2) := (\partial_{\omega_1}\Phi_1, \partial_{\omega_1}\Phi_2)|_{(\omega_1,\omega_2)=(\omega,\omega)}$, we obtain for $v \in H^1(\Gamma)$

$$\int_\Gamma \Big\{h_1'\overline{v}' + \omega h_1\overline{v} - (2p-1+b)(\Phi_0^\gamma)^{2p-2}h_1\overline{v} - bp(\Phi_0^\gamma)^{p-2}h_2\overline{v}\Big\}dx - \gamma h_{11}(0)\overline{v}_1(0) = -\int_\Gamma \Phi_0^\gamma\overline{v}dx$$
$$\int_\Gamma \Big\{h_2'\overline{v}' + \omega h_2\overline{v} - (2p-1+b(p-1))(\Phi_0^\gamma)^{2p-2}h_2\overline{v} - bp(\Phi_0^\gamma)^{p-2}h_1\overline{v}\Big\}dx - \gamma h_{21}(0)\overline{v}_1(0) = 0.$$

Analogous system can be obtained when differentiating with ω_2. Then setting $h_+ = h_1+h_2$ and $h_- = h_1-h_2$, we have

$$\begin{cases} \mathfrak{t}_+(h_+,v) = -(\Phi_0^\gamma, v)_2, \\ \mathfrak{t}_-(h_-,v) = -(\Phi_0^\gamma, v)_2, \end{cases} \tag{3.20}$$

where $\mathfrak{t}_+$ and $\mathfrak{t}_-$ are bilinear forms associated with the operators $\tilde{L}_+^R$ and $\tilde{L}_-^R$ defined by (3.13). Hence

$$d''(\omega) = \frac{1}{4}\begin{pmatrix} (\Phi_0^\gamma, h_+ + h_-)_2 & (\Phi_0^\gamma, h_+ - h_-)_2 \\ (\Phi_0^\gamma, h_+ - h_-)_2 & (\Phi_0^\gamma, h_+ + h_-)_2 \end{pmatrix}. \tag{3.21}$$

Observe that $\det(d''(\omega)) = \frac{(\Phi_0^\gamma,h_+)_2(\Phi_0^\gamma,h_-)_2}{2}$ and $\mathrm{trace}(d''(\omega)) = \frac{(\Phi_0^\gamma,h_+ + h_1)_2}{2}$.

Step 2. Below we will prove:

a) if $b > -1$, then $(\Phi_0^\gamma, h_+)_2 > 0$ for $2 \le p < 3$, and $(\Phi_0^\gamma, h_+)_2 < 0$ for $p > 3$;

b) if $b > p-1$, then $(\Phi_0^\gamma, h_-)_2 < 0$, and if $0 < b < p-1$, then $(\Phi_0^\gamma, h_-)_2 > 0$.

Firstly, we prove *a*). Since Φ_0^γ satisfies (3.9) and $v \in H^1(\Gamma)$ in (3.20) is arbitrary, we conclude that $h_+ = \partial_\omega\Phi_0^\gamma$. Then $(\Phi_0^\gamma, h_+)_2 = (\Phi_0^\gamma, \partial_\omega\Phi_0^\gamma)_2 = \frac{1}{2}\partial_\omega(\Phi_0^\gamma, \Phi_0^\gamma)_2$. Moreover, using (3.16), we have

$$\begin{aligned}\int_\Gamma (\Phi_0^\gamma)^2(x)dx &= \left(\frac{\omega}{b+1}\right)^{1/(p-1)}\int_\Gamma (\Psi_0^\gamma)^2(\sqrt{\omega}x)dx \\ &= \frac{\omega^{(3-p)/(2p-2)}}{(b+1)^{1/(p-1)}}\int_\Gamma (\Psi_0^\gamma)^2(x)dx,\end{aligned}$$

and then

$$\partial_\omega(\Phi_0^\gamma, \Phi_0^\gamma)_2 = \frac{(3-p)}{(2p-2)}\frac{\omega^{(5-3p)/(2p-2)}}{(b+1)^{1/(p-1)}}\int_\Gamma (\Psi_0^\gamma)^2(x)dx$$

which is positive for $2 \le p < 3$ and negative for $p > 3$. This proves *a*).

Secondly, we prove *b*) for $b > p-1$. Notice that the second line of (3.20) is equivalent to

$$\begin{aligned}&\int_\Gamma \Big\{h_-'(x)\overline{v}'(x) + \omega h_-(x)\overline{v}(x) - \frac{\omega(2p-b-1)}{b+1}\big(\Psi_0^\gamma(\sqrt{\omega}x)\big)^{2p-2}h_-(x)\overline{v}(x)\Big\}dx \\ &- \gamma h_{-,1}(0)\overline{v}_1(0) = -\left(\frac{\omega}{b+1}\right)^{1/2(p-1)}\int_\Gamma \Psi_0^\gamma(\sqrt{\omega}x)\overline{v}(x)dx.\end{aligned} \tag{3.22}$$

Denoting $f(\sqrt{\omega}x) = v(x), s(\sqrt{\omega}x) = h_-(x)$, from (3.22) we get

$$\begin{aligned}&\int_\Gamma \Big\{s'(y)\overline{f}'(y) + s(y)\overline{f}(y) - \frac{(2p-b-1)}{b+1}\big(\Psi_0^\gamma(y)\big)^{2p-2}s(y)\overline{f}(y)\Big\}dy - \frac{\gamma}{\sqrt{\omega}}s(0)\overline{f}_1(0) \\ &= -\frac{\omega^{(3-2p)/(2p-2)}}{(b+1)^{1/(2p-2)}}\int_\Gamma \Psi_0^\gamma(y)\overline{f}(y)dy.\end{aligned} \tag{3.23}$$

Moreover,

$$\int_\Gamma h_-(x)\,\Psi_0^\gamma(\sqrt{\omega}x)dx = \frac{1}{\sqrt{\omega}}\int_\Gamma h_-(\frac{y}{\sqrt{\omega}})\,\Psi_0^\gamma(y)dy = \frac{1}{\sqrt{\omega}}\int_\Gamma s(y)\,\Psi_0^\gamma(y)dy,$$

then $(h_-, \Phi_0^\gamma)_2$ and $(s, \Psi_0^\gamma)_2$ have the same sign. Observe that (3.23) can be rewritten as $\mathfrak{t}_\varepsilon(s,f) = -C(\omega)(\Psi_0^\gamma, f)_2$, where $\mathfrak{t}_\varepsilon$ is the bilinear form associated with self-adjoint operator $\tilde{L}_\varepsilon$ given by (3.17) (for $\varepsilon = \frac{2p-b-1}{b+1}$) and $C(\omega) = \frac{\omega^{(3-2p)/(2p-2)}}{(b+1)^{1/(2p-2)}}$. Set $f = s$, then by Lemma 3.13(i), we conclude that $0 < \mathfrak{t}_\varepsilon(s,s) = -C(\omega)(\Psi_0^\gamma, s)_2$ for $b > p-1$.

Thirdly, we prove b) for $0 < b < p-1$. Notice that from (3.23), by the Representation Theorem [27, Chapter VI, Theorem 2.1], we conclude

$$s \in \mathrm{dom}(\tilde{L}_\varepsilon), \qquad \tilde{L}_\varepsilon s = -C(\omega)\,\Psi_0^\gamma. \tag{3.24}$$

Since $\tilde{L}_\varepsilon$ is invertible for $1 < \varepsilon = \frac{2p-b-1}{b+1} < 2p-1$ (or equivalently for $0 < b < p-1$) and holomorphic in ε, then the solution $s = s(\varepsilon)$ of (3.24) is smooth in ε. Differentiating $\mathfrak{t}_\varepsilon(s(\varepsilon),\cdot)$ with ε, from (3.24) we obtain

$$\mathfrak{t}_\varepsilon\big(\partial_\varepsilon s(\varepsilon), f\big) = \big(s(\varepsilon)(\Psi_0^\gamma)^{2p-2}, f\big)_2.$$

Let $f = s(\varepsilon)$, then again, by (3.24),

$$\begin{aligned}0 &< \int_\Gamma |s(\varepsilon)|^2(\Psi_0^\gamma(x))^{2p-2}dx = \mathfrak{t}_\varepsilon\big(\partial_\varepsilon s(\varepsilon), s(\varepsilon)\big) = \big(\partial_\varepsilon s(\varepsilon), \tilde{L}_\varepsilon s(\varepsilon)\big)_2\\ &= -C(\omega)\big(\partial_\varepsilon s(\varepsilon), \Psi_0^\gamma\big)_2 = -C(\omega)\partial_\varepsilon\big(s(\varepsilon), \Psi_0^\gamma\big)_2.\end{aligned}$$

Hence $(s(\varepsilon), \Psi_0^\gamma)_2$ is decreasing.

It is easily seen that $v(\omega,x) = \omega^{1/(2p-2)}\,\Psi_0^\gamma(\sqrt{\omega}x)$ satisfies

$$-\Delta_\gamma v(\omega,x) + \omega v(\omega,x) - v(\omega,x)^{2p-1} = 0.$$

Further, from the above one concludes that $u(x) = \partial_\omega v(\omega,x)|_{\omega=1}$ is the solution to

$$-\Delta_\gamma u(x) + u(x) - (2p-1)\big(\Psi_0^\gamma(x)\big)^{2p-2}u(x) = -\,\Psi_0^\gamma(x).$$

Observe that $\partial_\omega v(\omega,x) \in \mathrm{dom}(\Delta_\gamma)$. Indeed, define

$$\hat{S}_\omega(\psi) = \frac{1}{2}\left\{\|\psi'\|_2^2 - \gamma|\psi_1(0)|^2 + \omega\|\psi\|_2^2\right\} - \frac{1}{2p}\|\psi\|_{2p}^{2p},$$

then $\hat{S}'_\omega(v(\omega,x)) = 0$ and therefore $\left\langle \hat{S}''_\omega(v(\omega,x))\partial_\omega v(\omega,x), g\right\rangle_{(H^1)^*\times H^1} = 0$, $g \in H^1(\Gamma)$, which yields

$$\begin{aligned}&\int_\Gamma (\partial_\omega v(\omega,x))'\overline{g}(x)dx - \gamma\partial_\omega v_1(\omega,0)\overline{g}_1(0)\\ &= \int_\Gamma \Big\{-\omega\partial_\omega v(\omega,x)\overline{g}(x) + \Big(\partial_\omega v(\omega,x) + (2p-2)\,\mathrm{Re}(\partial_\omega v(\omega,x))\Big)\omega(\Psi_0^\gamma(\sqrt{\omega}x))^{2p-2}\overline{g}(x)\Big\}\,dx.\end{aligned}$$

By the Representation Theorem, $\partial_\omega v(\omega,x) \in \mathrm{dom}(\Delta_\gamma)$.

It is easily seen that $\partial_\omega v(\omega,x)|_{\omega=1}C(\omega) = s(2p-1)$, where $s(2p-1)$ is the solution to (3.24) for $\varepsilon = 2p-1$. By the continuity of the scalar product, we conclude

$$\begin{aligned}&\big(s(2p-1), \Psi_0^\gamma\big)_2 = \big(\partial_\omega v(\omega,x)|_{\omega=1}C(\omega), \Psi_0^\gamma\big)_2 = C(\omega)\frac{1}{2}\partial_\omega\|v(\omega,x)\|^2|_{\omega=1}\\ &= \frac{1}{2}C(\omega)\partial_\omega\Big(\omega^{(3-p)/(2p-2)}\int_\Gamma(\Psi_0^\gamma)^2dx\Big)\Big|_{\omega=1} > 0 \quad \text{for} \quad p < 3.\end{aligned}$$

Finally, since $(s(\varepsilon), \Psi_0^\gamma)_2$ is decreasing on the interval $(1, 2p-1)$, we get $(s(\varepsilon), \Psi_0^\gamma)_2 > 0$ for $\varepsilon \in (1, 2p-1)$.

Step 3. Let $\gamma > 0, k = 0$. By (3.15), Lemmas 3.12 and 3.14, we have $n(\tilde{L}^R) = 1$ for $b > p-1$, and $n(\tilde{L}^R) = 2$ for $0 < b < p-1$.

Let $2 \le p < 3$, then, by *Step 2* and (3.21), we conclude that $p(d''(\omega)) = 1$ for $b > p-1$, and $p(d''(\omega)) = 2$ for $0 < b < p-1$. Let $p > 3$ and $b > p-1$, then $p(d''(\omega)) = 0$.

Finally, for $2 \le p < 3$ and $b \ne p-1$, we get $n(\tilde{L}) = p(d''(\omega))$, therefore, $e^{i\omega t}(\Phi_0^\gamma, \Phi_0^\gamma)$ is orbitally stable. For $p > 3$ and $b > p-1$, we obtain $n(\tilde{L}) - p(d''(\omega)) = 1$, then, by [23, Theorem 5.1], there exists positive λ satisfying (3.18), and orbital instability follows by C^2 regularity of the data-solution mapping associated with (3.8). □

3.3. Stability of the standing wave generated by 2D rotation

Observe that for $p = 2$ and $b = 1$, in addition to the symmetry $T_0(\theta_0)(u, v) = e^{i\theta_0}(u, v), \theta_0 \in \mathbb{R}$, system (3.8) is also invariant under 2D rotation $T_1(\theta_1)$:

$$\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} \overset{T_1(\theta_1)}{\mapsto} \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix}, \quad \theta_1 \in \mathbb{R}.$$

Below we study orbital stability of the standing wave related to both of these symmetries:

$$(u(t,x), v(t,x)) = T_0(\omega_0 t) T_1(\omega_1 t) \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix}, \quad \omega_0, \omega_1 \in \mathbb{R}^+.$$

It is easily seen that $T_1'(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Assume that the profile of the standing wave has the form $(\Phi_1, \Phi_2) = \Phi_{\omega_0,\omega_1}(x) \cdot (v_1, v_2)$, where $(v_1, v_2) = (\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is the unit eigenvector of $iT_1'(0)$ corresponding to the eigenvalue 1. Then the profile $\Phi_{\omega_0,\omega_1}(x)$ satisfies the stationary equation

$$-\Delta_\gamma \Phi + (\omega_0 - \omega_1)\Phi - |\Phi|^2 \Phi = 0. \tag{3.25}$$

The description of solutions is given by (3.10) (with ω substituted by $\omega_0 - \omega_1$). Below we will prove the following theorem.

Theorem 3.15. *Let $\omega_0 - \omega_1 > \frac{\gamma^2}{N^2}$ and Φ_k^γ be defined by (3.10), then for $k = 0, \gamma > 0$ the standing wave $T_0(\omega_0 t)T_1(\omega_1 t)\binom{i\Phi_k^\gamma/\sqrt{2}}{\Phi_k^\gamma/\sqrt{2}}$ is orbitally stable. Moreover, if either $\gamma > 0$ and k is odd, or $\gamma < 0$ and $N-k-1$ is odd, then $T_0(\omega_0 t)T_1(\omega_1 t)\binom{i\Phi_k^\gamma/\sqrt{2}}{\Phi_k^\gamma/\sqrt{2}}$ is spectrally unstable.*

In the above theorem the stability is understood in the sense of the $U(1)$-symmetry. This is related to the fact that the proof uses [23, Stability Theorem and Instability Theorem] where the stability/instability results are stated for a centralizer subgroup.

Conserved functional generated by $T_1'(0)$ is given by

$$Q_1(u, v) = \operatorname{Im} \int_\Gamma u\bar{v}dx,$$

and the standing wave $T_0(\omega_0 t)T_1(\omega_1 t)\binom{i\Phi_k^\gamma/\sqrt{2}}{\Phi_k^\gamma/\sqrt{2}}$ is a critical point of the functional

$$S_{\omega_0,\omega_1}(u, v) = \frac{1}{2}\left\{E(u, v) + \omega_0(\|u\|_2^2 + \|v\|_2^2)\right\} + \omega_1 Q_1(u, v).$$

As in the previous subsections, we associate with the bilinear form $\left\langle S''_{\omega_0,\omega_1}\left(\frac{i\Phi_k^\gamma}{\sqrt{2}},\frac{\Phi_k^\gamma}{\sqrt{2}}\right)(h_1,h_2),(z_1,z_2)\right\rangle_{X^*\times X}$ the self-adjoint in $L^2(\Gamma)\times L^2(\Gamma)$ operator $\tilde{L}$. Let $h_j = h_j^R + i h_j^I$, $j=1,2$. Substituting complex-valued vector function $\vec{h}=(h_1,h_2)$ by the corresponding quadruplet of real-valued functions $(h_1^R, h_1^I, h_2^R, h_2^I)$ and $\tilde{L}\vec{h}=(f,g)=(f^R+if^I, g^R+ig^I)$ by the quadruplet (f^R, f^I, g^R, g^I), we can express the action of $\tilde{L}$ as

$$\begin{pmatrix} f^R \\ f^I \\ g^R \\ g^I \end{pmatrix} = \begin{pmatrix} L_{\omega_0}-(\Phi_k^\gamma)^2 & 0 & 0 & \omega_1 \\ 0 & L_{\omega_0}-2(\Phi_k^\gamma)^2 & -\omega_1-(\Phi_k^\gamma)^2 & 0 \\ 0 & -\omega_1-(\Phi_k^\gamma)^2 & L_{\omega_0}-2(\Phi_k^\gamma)^2 & 0 \\ \omega_1 & 0 & 0 & L_{\omega_0}-(\Phi_k^\gamma)^2 \end{pmatrix} \begin{pmatrix} h_1^R \\ h_1^I \\ h_2^R \\ h_2^I \end{pmatrix}, \tag{3.26}$$

where $L_{\omega_0} = -\Delta_\gamma + \omega_0$ with $\mathrm{dom}(L_{\omega_0}) = \mathrm{dom}(\Delta_\gamma)$. Below we characterize the spectral properties of the operator $\tilde{L}$.

Lemma 3.16. *Let $\omega_0-\omega_1 > \frac{\gamma^2}{N^2}$ and Φ_{ω_0,ω_1} be the solution to (3.25) given by (3.10) (with ω substituted by $\omega_0-\omega_1$). Let also operator $\tilde{L}$ be defined by (3.26). Then the following assertions hold:*

(i) $\ker(\tilde{L}) = \mathrm{span}\{(\Phi_k^\gamma, -i\Phi_k^\gamma)\}$;

(ii) $n(\tilde{L}) = \begin{cases} k+1, & \gamma>0 \\ N-k, & \gamma<0 \end{cases}$;

(iii) $\sigma_{\mathrm{ess}}(\tilde{L})>0$.

Proof. Observe that $(\tilde{L}-\lambda)\vec{h}=(f,g)$ is equivalent to

$$\begin{cases} (L_{\omega_0}+\omega_1-(\Phi_k^\gamma)^2-\lambda)(h_1^R+h_2^I) = f^R+g^I \\ (L_{\omega_0}-\omega_1-(\Phi_k^\gamma)^2-\lambda)(h_1^R-h_2^I) = f^R-g^I \\ (L_{\omega_0}-\omega_1-3(\Phi_k^\gamma)^2-\lambda)(h_1^I+h_2^R) = f^I+g^R \\ (L_{\omega_0}+\omega_1-(\Phi_k^\gamma)^2-\lambda)(h_1^I-h_2^R) = f^I-g^R. \end{cases} \tag{3.27}$$

By [3, Proposition 6.1], the operator $L_{\omega_0}-\omega_1-(\Phi_k^\gamma)^2$ is nonnegative and $\ker(L_{\omega_0}-\omega_1-(\Phi_k^\gamma)^2)=\mathrm{span}\{\Phi_k^\gamma\}$. Moreover, $\ker(L_{\omega_0}-\omega_1-3(\Phi_k^\gamma)^2)=\{0\}$ and, by [26, Theorem 3.1], $n(L_{\omega_0}-\omega_1-3(\Phi_k^\gamma)^2) = \begin{cases} k+1, & \gamma>0 \\ N-k, & \gamma<0 \end{cases}$. Observing that operator $L_{\omega_0}+\omega_1-(\Phi_k^\gamma)^2$ is positive definite, we arrive at (i), (ii).

Finally, noticing that for $\lambda\in\mathbb{R}^+\setminus[\omega_0-\omega_1,\infty)$ all the operators on the left side of (3.27) are invertible, we conclude that $\lambda\in\rho(\tilde{L})$ (since $\tilde{L}-\lambda$ is bijective), and therefore (iii) follows. □

Proof of Theorem 3.15. Without abuse of notation, we will write S''_{ω_0,ω_1} instead of $S''_{\omega_0,\omega_1}\left(\frac{i\Phi_0^\gamma}{\sqrt{2}},\frac{\Phi_0^\gamma}{\sqrt{2}}\right)$. Observe that S''_{ω_0,ω_1} satisfies [45, Condition (G)]. Indeed, for $\vec{v}=(v_1,v_2)\in X$

$$\begin{aligned} \langle S''_{\omega_0,\omega_1}\vec{v},\vec{v}\rangle_{X^*\times X} &= \int_\Gamma \Big\{|v_1'|^2+|v_2'|^2+\omega_0\Big(|v_1|^2+|v_2|^2\Big)\Big\}dx \\ &- \int_\Gamma \Big\{(\Phi_0^\gamma)^2\Big(|v_1|^2+|v_2|^2+(\mathrm{Re}\,v_1)^2+(\mathrm{Re}\,v_2)^2+2\,\mathrm{Re}\,v_2\,\mathrm{Im}\,v_1\Big)\Big\}dx \\ &- 2\omega_1\,\mathrm{Im}\int_\Gamma v_1\overline{v_2}dx - \gamma\Big(|v_{11}(0)|^2+|v_{21}(0)|^2\Big) \\ &\geq \min\{\tfrac{1}{2},\omega_0\}\|\vec{v}\|_X^2 - \Big(3M^2+\frac{2\gamma^2}{N^2}+\omega_1\Big)\|\vec{v}\|_2^2, \end{aligned}$$

where $M=\|\Phi_0^\gamma\|_\infty$. Thus, by [45, Lemma 5.4] and Lemma 3.16, we conclude

$$\sigma_{\mathrm{ess}}(\mathcal{R}^{-1}S''_{\omega_0,\omega_1}) = \sigma_{\mathrm{ess}}(\tilde{L})>0,\quad \ker(\mathcal{R}^{-1}S''_{\omega_0,\omega_1}) = \mathrm{span}\{(\Phi_k^\gamma,-i\Phi_k^\gamma)\},$$
$$n(\mathcal{R}^{-1}S''_{\omega_0,\omega_1}) = n(\tilde{L}).$$

Since the centralizer subgroup of the group $\{T_0(\theta_0)T_1(\theta_1) : \theta_0, \theta_1 \in \mathbb{R}\}$ coincides with $G_{\theta_0,\theta_1} = \{T_0(\theta) : \theta \in \mathbb{R}\}$, then according to [23, Stability Theorem and Instability Theorem], we can analyze stability only in context of the $U(1)$-symmetry. It is easily seen that *Assumptions 1–3* of [23] are satisfied (in particular, *Assumption 3* is satisfied due to Lemma 3.16).

Let $d''(\omega_0 - \omega_1)$ be the reduced Hessian of $S_{\omega_0,\omega_1}\left(\frac{i\Phi_k^\gamma}{\sqrt{2}}, \frac{\Phi_k^\gamma}{\sqrt{2}}\right)$. Thus, by [23, Stability Theorem and Instability Theorem], we conclude:

- if $n(\tilde{L}) = p(d''(\omega_0 - \omega_1))$, then $T_0(\omega_0 t)T_1(\omega_1 t)\binom{i\Phi_k^\gamma/\sqrt{2}}{\Phi_k^\gamma/\sqrt{2}}$ is orbitally stable;
- if $n(\tilde{L}) - p(d''(\omega_0 - \omega_1))$ is odd, then $T_0(\omega_0 t)T_1(\omega_1 t)\binom{i\Phi_k^\gamma/\sqrt{2}}{\Phi_k^\gamma/\sqrt{2}}$ is spectrally unstable. Using [7, Proposition 5], we conclude $d''(\omega_0 - \omega_1) = \frac{1}{2}\partial_{\omega_0}\|\Phi_k^\gamma\|_2^2 > 0$. Finally, applying Lemma 3.16(*ii*), we arrive at the result. □

Appendix

In this Appendix we recall some basic properties of symmetric rearrangements of a measurable function $u : \Gamma \to \mathbb{C}^N$. Set

$$\mu_u(s) = |\{x \in \Gamma : |u(x)| \geq s\}| \quad \text{and} \quad \lambda_u(t) = \sup\{s : \mu_u(s) > Nt\}.$$

Here $\{x \in \Gamma : |u(x)| \geq s\} = \bigcup_{e=1}^N \{x \in I_e : |u_e(x)| \geq s\}$, and all sets in the union are disjoint. The symmetric rearrangement u^* of u is defined by $u^* = (u_e^*)_{e=1}^N$ with $u_e^* = \lambda_u$ for all $e = 1, \ldots, N$. Basic properties of symmetric rearrangements are listed in the next proposition.

Proposition A.17. *Let $u, v \in H^1(\Gamma)$. Then the following assertions hold.*
(i) The symmetric rearrangement u^* is positive and nonincreasing. Moreover, u and u^* are equimeasurable, that is, for every $s > 0$

$$|\{|u| \geq s\}| = |\{u^* \geq s\}|. \tag{A.28}$$

(ii) $u^* \in H^1(\Gamma)$, $\|u\|_p = \|u^*\|_p$ for all $1 \leq p \leq \infty$, and

$$\left\|(u^*)'\right\|_2 \leq \frac{N}{2}\|u'\|_2. \tag{A.29}$$

(iii) If u, v are nonnegative, then

$$(u, v)_2 \leq (u^*, v^*)_2.$$

Proof. The proof of statements (*i*) and (*ii*) is contained in [3, Proposition A.1, Theorem 6]. It follows from the Layer cake representation theorem [30, Theorem 1.13] and (A.28) that

$$\begin{aligned}
&\int_\Gamma u(x)v(x)dx = \sum_{e=1}^N \int_{\mathbb{R}^+} u_e(x)v_e(x)dx = \sum_{e=1}^N \int_{\mathbb{R}^+} \left(\int_0^\infty \chi_{\{u_e \geq t\}}(x)dt \int_0^\infty \chi_{\{v_e \geq s\}}(x)ds\right)dx \\
&= \sum_{e=1}^N \int_0^\infty \int_0^\infty |\{u_e \geq s\} \cap \{v_e \geq t\}|\, dsdt = \int_0^\infty \int_0^\infty |\{u \geq s\} \cap \{v \geq t\}|\, dsdt \\
&\leq \int_0^\infty \int_0^\infty \min\{|\{u \geq s\}|\,;\,|\{v \geq t\}|\}\, dsdt = \int_0^\infty \int_0^\infty \min\{|\{u^* \geq s\}|\,;\,|\{v^* \geq t\}|\}\, dsdt \\
&= \int_0^\infty \int_0^\infty |\{u^* \geq s\} \cap \{v^* \geq t\}|\, dsdt = \int_\Gamma u^*(x)v^*(x)dx.
\end{aligned}$$

Hence (*iii*) holds. □

References

[1] R. Adami, C. Cacciapuoti, D. Finco, D. Noja, On the structure of critical energy levels for the cubic focusing NLS on star graphs, J. Phys. A 45 (192001) (2012) 7.
[2] R. Adami, C. Cacciapuoti, D. Finco, D. Noja, Constrained energy minimization and orbital stability for the NLS equation on a star graph, Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (6) (2014) 1289–1310.
[3] R. Adami, C. Cacciapuoti, D. Finco, D. Noja, Variational properties and orbital stability of standing waves for NLS equation on a star graph, J. Differential Equations 257 (2014) 3738–3777.
[4] R. Adami, C. Cacciapuoti, D. Finco, D. Noja, Stable standing waves for a NLS on star graphs as local minimizers of the constrained energy, J. Differential Equations 260 (10) (2016) 7397–7415.
[5] N.N. Akhmediev, A. Ankiewicz, Solitons, Nonlinear Pulses and Beams, Chapman & Hall, London, 1997.
[6] J. Angulo, N. Goloshchapova, Extension theory approach in the stability of the standing waves for the NLS equation with point interactions on a star graph, Adv. Differential Equations 23 (11–12) (2018) 793–846.
[7] J. Angulo, N. Goloshchapova, On the orbital instability of excited states for the NLS equation with the δ-interaction on a star graph, Discrete Contin. Dyn. Syst. 38 (10) (2018) 5039–5066.
[8] A.H. Ardila, L. Cely, N. Goloshchapova, Instability of ground states for the NLS equation with potential on the star graph, J. Evol. Equ. (21) (2021) 3703–3732.
[9] T. Bartsch, Z.-Q. Wang, Note on ground states of nonlinear Schrödinger systems, J. Partial Differ. Equ. 19 (2006) 200–207.
[10] G. Beck, S. Imperiale, P. Joly, Mathematical modelling of multi conductor cables, Discrete Contin. Dyn. Syst. Ser. S. 8 (2015) 521–546.
[11] G. Berkolaiko, C. Carlson, S. Fulling, P. Kuchment, Quantum graphs and their applications, in: Contemporary Math. Vol. 415, Amer. Math. Soc., Providence, RI, 2006.
[12] S. Bhattarai, Stability of solitary-wave solutions of coupled NLS equations with power-type nonlinearities, Adv. Nonlinear Anal. 4 (2) (2015) 73–90.
[13] C. Cacciapuoti, D. Finco, D. Noja, Ground state and orbital stability for the NLS equation on a general starlike graph with potentials, Nonlinearity 30 (2017) 3271–3303.
[14] T. Cazenave, Semilinear Schrödinger equations, in: Courant Lect. Notes in Math. Vol. 10, New York University, Courant Institute of Mathematical Sciences, New York, Amer. Math. Soc., Providence, RI, 2003.
[15] R. Cipolatti, W. Zumpichiatti, Orbitally stable standing waves for a system of coupled nonlinear Schrödinger equations, Nonlinear Anal. 42 (2000) 445–461.
[16] P. Exner, J.P. Keating, P. Kuchment, T. Sunada, A. Teplyaev, Analysis on graphs and its applications, in: Proc. Symp. Pure Math. Vol. 77, Amer. Math. Soc., Providence, RI, 2008.
[17] R. Fukuizumi, L. Jeanjean, Stability of standing waves for a nonlinear Schrödinger equation with a repulsive Dirac delta potential, Discrete Contin. Dyn. Syst. 21 (2008) 121–136.
[18] S. Gnutzman, U. Smilansky, S. Derevyanko, Stationary scattering from a nonlinear network, Phys. Rev. A 83 (033831) (2011).
[19] N. Goloshchapova, Dynamical and variational properties of NLS-δ_s' equation on the star graph, J. Differential Equations 310 (2022) 1–44.
[20] N. Goloshchapova, M. Ohta, Blow-up and strong instability of standing waves for the NLS-δ equation on a star graph, Nonlinear Anal. 196 (111753) (2020).
[21] M. Grillakis, Linearized instability for nonlinear schrödinger and Klein–Gordon equations, Comm. Pure Appl. Math. 41 (1988) 747–774.
[22] M. Grillakis, J. Shatah, W. Strauss, Stability theory of solitary waves in the presence of symmetry I, J. Funct. Anal. 74 (1) (1987) 160–197.
[23] M. Grillakis, J. Shatah, W. Strauss, Stability theory of solitary waves in the presence of symmetry II, J. Funct. Anal. 94 (1990) 308–348.
[24] D. Henry, J.F. Perez, W. Wreszinski, Stability theory for solitary-wave solutions of scalar field equation, Comm. Math. Phys. 85 (1982) 351–361.
[25] P. Joly, A. Semin, Mathematical and numerical modeling of wave propagation in fractal trees, C.R. Math. Acad. Sci. Paris 349 (2011) 1047–1051.
[26] A. Kairzhan, Orbital instability of standing waves for NLS equation on star graphs, Proc. Amer. Math. Soc. 147 (7) (2019) 2911–2924.
[27] T. Kato, Perturbation theory for linear operators, in: Die Grundlehren Der Mathematischen Wissenschaften, Band Vol. 132, Springer-Verlag New York, Inc., New York, 1966.
[28] Y.S. Kivshar, G.P. Agrawal, Optical Solitons: From Fibers to Photonic Crystals, Academic Press, San Diego, 2003.
[29] P. Kuchment, Graph models for waves in thin structures, Waves Random Media 12 (2002) R1–R24.
[30] E. Lieb, M. Loss, Analysis, Volume 14 of Graduate Studies in Mathematics, second ed., American Mathematical Society, Providence, RI, 2001.
[31] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. I, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 109–145.
[32] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. II, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 223–283.
[33] O. Lopes, Stability of solitary waves of some coupled systems, Nonlinearity 19 (1) (2006) 9–113.
[34] O. Lopes, Stability of solitary waves for a generalized nonlinear coupled Schrödinger systems, São Paulo J. Math. Sci. 5 (2) (2011) 175–184.

[35] L.A. Maia, E. Montefusco, B. Pelacci, Orbital stability property for coupled nonlinear Schrödinger equations, Adv. Nonlinear Stud. 10 (3) (2010) 681–705.
[36] L.A. Maia, E. Montefusco, B. Pellacci, Positive solutions for a weakly coupled non linear Schrödinger system, J. Differential Equations 229 (2006) 743–767.
[37] S.V. Manakov, On the theory of two-dimensional stationary self-focusing of electromagnetic waves, Sov. Phys.—JETP 38 (1974) 248–253.
[38] C.R. Menyuk, Nonlinear pulse propagation in birefringent optical fibers, IEEE J. Quantum Electron. 23 (1987) 174–176.
[39] A.E. Miroshnichenko, M.I. Molina, Y.S. Kivshar, Localized modes and bistable scattering innonlinear network junctions, Phys. Rev. Lett. 75 (04602) (2007).
[40] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, Inequalities involving functions and their integrals and derivatives, Math. Appl. 53 (1991).
[41] N.V. Nguyen, Z.-Q. Wang, Orbital stability of solitary waves for a nonlinear Schrödinger system, Adv. Differential Equations 16 (9–10) (2011) 977–1000.
[42] D. Noja, Nonlinear Schördinger equation on graphs: recent results and open problems, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 372 (2007) (2014) 20130002.
[43] D.E. Pelinovsky, J. Yang, Instabilities of multi-hump vector solitons in coupled nonlinear Schrödinger equations, Stud. Appl. Math. 115 (1) (2005) 109–137.
[44] X. Song, Stability and instability of standing waves to a system of Schröodinger equations with combined power-type nonlinearities, J. Math. Anal. Appl. 366 (1) (2010) 345–359.
[45] C.A. Stuart, Lectures on the orbital stability of standing waves and application to the nonlinear Schrödinger equation, Milan J. Math. 76 (2008) 329–399.
[46] A. Tokuno, M. Oshikawa, E. Demler, Dynamics of the one dimensional Bose liquids: Andreev-like reflection at Y-junctions and the absence of Aharonov-Bohmeffect, Phys. Rev. Lett. 100 (140402) (2008).
[47] M. Wadati, T. Iizuka, M. Hisakado, A coupled nonlinear Schrödinger equation and optical solitons, J. Phys. Soc. Japan 61 (7) (1992) 2241–2245.
[48] J. Wei, W. Yao, Uniqueness of positive solutions for some coupled nonlinear Schrödinger equations, Commun. Pure Appl. Anal. 11 (3) (2012) 1003–1011.